Advances in Packing Directed Joins

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Abstract

Several of the finest unclaimed prizes in directed graph theory involve the packing of directed joins. One difficulty in claiming these prizes is that the broad conjecture posed by Edmonds and Giles, whether the maximum number of disjoint directed joins equals the smallest weight of a directed cut in every weighted directed graph, is not true in general. This is despite the fact that the conjecture is true in several special cases, and is also true if the roles of directed joins and directed cuts are reversed. Another difficulty is that the known minimal counterexamples, one found by Schrijver and two found by Cornuéjols and Guenin during a computer search, are mysterious in nature. We dispel some of this mystery by providing a framework for understanding the known counterexamples. We then use this framework to construct several new counterexamples, and to prove that every "smallest" minimal counterexample has now been found. Finally, we temper these advances by introducing an NP-completeness result for a more difficult problem.

Keywords: directed graph, directed cut, directed join, digraph, dicut, dijoin, Edmonds-Giles Conjecture, Lucchesi-Younger Theorem, NP-complete, min-max

1 Introduction

Let $D = (N, A)$ be a directed graph, with arc a, arc subset B, and non-empty strict subset of nodes M. The *cut induced by* M, denoted $\delta(M)$, is the set of arcs directed from a node in M, to a node in $N - M$. The cut $\delta(M)$ is *directed* if $\delta(N - M)$ is empty. Every arc must be present in either a directed cut, or a directed cycle, but not both. Let D/B be the directed graph obtained from D by contracting each arc in B. Notice that $\delta(M)$ is a directed cut in $D/\{a\}$, if and only if, $\delta(M)$ is a directed cut in D without a in $\delta(M)$. If D has no directed cuts, then D is *strongly connected*; if D/B is strongly connected, then B is a *directed join* in D. From this discussion, B is a directed join in D, if and only if, B has a non-empty intersection with each directed cut in D.

Directed graph D can be *extended* to the weighted directed graph (D, ω), if the vector ω assigns a non-negative integer to each of its arcs. The *weight* of a is $\omega(a)$, and the *weight* of B, denoted $\omega(B)$, is the sum of the weights of the arcs in B. Let $\tau(D, \omega)$ be the smallest weight of a directed cut in (D, ω). A collection of directed joins is *disjoint* unless the weight of some arc a is exceeded by the number of directed joins including a. Let $\nu(D, \omega)$ be the largest cardinality of a collection of disjoint directed joins in (D, ω). Notice that $\tau(D, \omega) \geq \nu(D, \omega)$.

Conjecture 1.1 (Edmonds-Giles [1]) *$\tau(D, \omega) = \nu(D, \omega)$ for every (D, ω).*

Using different techniques, Schrijver [2], Feofiloff [3], and Feofiloff and Younger [4] have proven a significant special case of Conjecture 1.1. A node s is a *source* or *sink*, if $\delta(\{s\})$ or $\delta(N - \{s\})$ is a directed cut, respectively. A directed graph is *source-sink connected* if directed paths connect each source to each sink.

Theorem 1.2 *If D is source-sink connected, then $\tau(D, \omega) = \nu(D, \omega)$.*

Schrijver [2] proved a stronger version of Theorem 1.2, and showed how this generalizes theorems of Menger, Gupta, and Edmonds. Conjecture 1.1 is also true when D is series-parallel [5], and when D is a directed tree together with every transitive arc [2]. Furthermore, if the roles of directed joins and directed cuts are reversed, then the result is true by the Lucchesi-Younger

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Theorem [6]. Algorithmically, [2], [3] and [4] find $\nu(D,\omega)$ disjoint directed joins in polynomial time when D is source-sink connected. Schrijver [7] asks if the same is possible for any weighted directed graph. Related open problems include bounding the gap between $\tau(D,\omega)$ and $\nu(D,\omega)$, and Woodall's Conjecture [8], which asks if $\tau(D,\omega) = \nu(D,\omega)$ when ω is positive.

Conjecture 1.1 can be restated as the following conjecture. Let Y be a set of directed cuts. If for all $\delta(M_1)$ and $\delta(M_2)$ in Y, the *union*, $\delta(M_1 \cup M_2)$, and *intersection*, $\delta(M_1 \cap M_2)$, are never directed cuts outside of Y, then Y is *closed*. A Y*-cover* is a set of arcs intersecting each directed cut in Y. A set of Y-covers is *disjoint* unless the weight of some arc is exceeded by the number of directed joins it is present in. Let $\tau(D,Y,\omega)$ be the smallest weight of a directed cut in Y, and $\nu(D,Y,\omega)$ be the largest cardinality of a set of disjoint Y-covers.

Conjecture 1.3 *If Y is closed, then $\tau(D,Y,\omega) = \nu(D,Y,\omega)$.*

Conjecture 1.1 implies Conjecture 1.3 by taking (D,ω) and Y, and adding zero weight arcs between pairs of vertices whenever the new the arc does not violate a directed cut in Y. The new directed graph has Y as its directed cuts. Moreover, the weight of each directed cuts has not changed. Similarly, Theorem 1.2 can be restated [4]. A directed cut, $\delta(M)$, is a *side cut*, if every source is in M, or no sink is in M. The set of side cuts is closed, and every directed cut in a source-sink connected directed graph is a side cut.

Theorem 1.4 *If Y is the set of side cuts, then $\tau(D,Y,\omega) = \nu(D,Y,\omega)$.*

Completing the list of minimal counterexamples to Conjecture 1.1 could result in even stronger min-max theorems. For example, a source s is *super* if there are directed paths from s to every sink, and a sink s is *super* if there are directed paths from every source to s. In a source-sink connected graph, all sources and sinks are super. On the other hand, Cornuéjols and Guenin [9] discovered a counterexample that contains a super source. In this sense, the following conjecture is the strongest possible generalization of Theorem 1.2

Conjecture 1.5 *If D has a super source and super sink, then $\tau(D,\omega) = \nu(D,\omega)$.*

Figure 1 illustrates the previously known minimal counterexamples [10] [9], where thin and thick arcs have weight zero and one, respectively. Part a) has (D_1,ω_1) due to Schrijver [7], b) and c) have (D_2,ω_2) and (D_3,ω_3) due to Cornuéjols and Guenin [9]. Part d) has a new embedding of (D_3,ω_3) reminiscent of (D_1,ω_1).

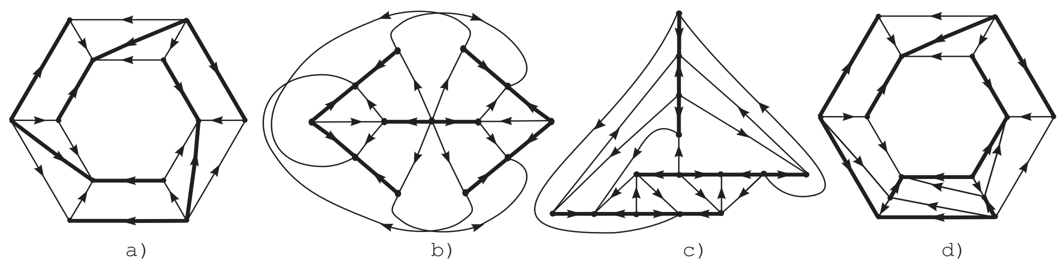

Fig. 1. Previously known minimal counterexamples with two embeddings of (D_3, ω_3)

2 Understanding the Known Counterexamples

By considering one endpoint of a path P to be the end of P, we may say that an arc in P is *forward* if it is directed towards the end of P; otherwise the arc is *backward*. Path P is *alternating* if all its consecutive arcs are oppositely directed. A node is *internal* to P if it is not at either end of P. An alternating path with weight one arcs is an *s-path*, if each of its internal nodes is a source or sink inducing a directed cut of weight two. In this section, suppose that $\tau(D, \omega)$ is two, and the non-zero weight arcs of (D, ω) partition uniquely into k s-paths called P_1 through P_k; notice that these properties are true for each (D_i, ω_i) in Figure 1. A subset of arcs J is a *crossing set* if J independently contains either the forward or backward arcs of each s-path. Hence, (D, ω) has 2^k crossing sets, and 2^{k-1} pairs of disjoint crossing sets. Given a crossing set J, assign a vector $b = b(J)$, where $b[i]$ is zero or one, depending on whether J contains the backward or forward arcs in P_i, respectively.

Lemma 2.1 *If J_1 and J_2 are disjoint directed joins in (D, ω), then J_1 and J_2 are a disjoint pair of crossing sets.*

Proof: The arcs on each P_i have weight one, and the internal nodes of each P_i induce directed cuts of weight two. Therefore, along each s-path, J_1 and J_2 each contain one arc out of every consecutive pair of arcs. This is only possible if J_1 and J_2 are disjoint crossing sets. □

To complement crossing sets, a directed cut is a *crossing cut* if it intersects each s-path at most once. The crossing cuts of (D_1, ω_1) are shown in Figure 2. For crossing cut $c = \delta(M)$, assign a vector $t = t(c)$ of length k, where $t[i]$ is zero, one, or $*$, depending on whether c intersects a backward arc, forward arc, or no arc in P_i. For a vector $t \in \{0, 1, *\}^k$, let $b = b(t)$ contain the binary vectors of length k such that $b[i]$ is zero if $t[i]$ is zero, and $b[i]$ is one if $t[i]$ is one. Hence, $b(t)$ contains 2^x binary vectors, where x is the number of $*$ entries in t. Finally, the *trace* of (D, ω), denoted $\mathbb{T}(D, \omega)$, is the union of $b(t(c))$ over all crossing cuts c in (D, ω). A trace $\mathbb{T}$ is *balanced* if for all k-bit binary vectors x, at least one of x and $\overline{x}$ is in $\mathbb{T}$, where $\overline{x}$ is the bit-wise complement of x.

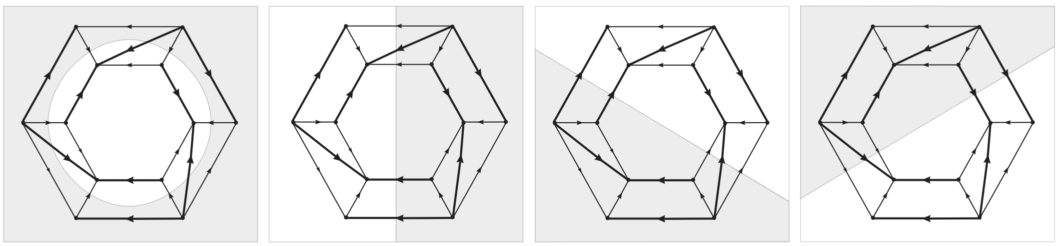

Fig. 2. The crossing cuts of (D_1, ω_1).

Lemma 2.2 *If J is a crossing set, then J intersects every directed cut in X, except those crossing cuts c with $b(J) \in b(t(c))$.*

Proof: By definition, J intersects crossing cut c, if and only if, $b(J) \notin b(t(c))$. Furthermore, if $\delta(M)$ intersects s-path P_i more than once, then $\delta(M)$ must intersect at least one backward and one forward arc of P_i, and is therefore intersected by J □

Corollary 2.3 *If $\mathbb{T}(D, \omega)$ is balanced then $\nu(D, \omega) = 1$ otherwise $\nu(D, \omega) = 2$*

Proof: This follows immediately from Lemmas 2.1 and 2.2. □

From Corollary 2.3, determining the value τ for (D_1, ω_1) through (D_3, ω_3) reduces to computing the trace. In each case, the reader may verify that $\mathbb{T}$ is balanced and is equal to $\{000, 011, 101, 110\}$. In Section 5 we show that this is the only possible trace for minimal counterexamples with $\tau = 2$ and $k = 3$.

3 Constructing New Counterexamples

Let us begin this section by recalling the notion of minimality used in [9]. Directed graph D' is *contractible* to D, if for some subset A' of arcs in D', we have D'/A' equal to D (up to deleting loop arcs). Arc $a = uw$ is *transitive* in D if D has a directed path from u to w that avoids a. D' is a *transitive extension* of D, if D' can be obtained from D by adding arcs that are transitive in D. Cornuéjols and Guenin [9] remarked that if D is contractible to a transitive extension of D_i, for i in $\{1, 2, 3\}$, then there exists ω such that $\tau(D, \omega)$ is two and $\nu(D, \omega)$ is one. (Use weight zero for the transitive arcs, weight $\tau = 2$ for the arcs to be contracted, and weight ω_i otherwise). It is natural to ask if this accounts for all counterexamples [9]; in other words, are (D_1, ω_1) through

(D_3, ω_3) the only minimal counterexamples? We negatively answer this question with three modifications that generate new minimal counterexamples.

Modification One: The first modification involves deleting combinations of weight zero arcs without decreasing τ. Since this operation cannot increase ν, the results are new counterexamples. Further counterexamples can then be created by adding non-transitive weight zero arcs, so long as these arcs would have been transitive in the original. Modifications of this type can be made to (D_2, ω_2) and (D_3, ω_3), and we let $(D_2, \omega_2)^+$ and $(D_3, \omega_3)^+$ represent the resulting families of minimal counterexamples. Figure 3 a) shows a member of $(D_3, \omega_3)^+$ that has had two arcs deleted, and then one arc t added.

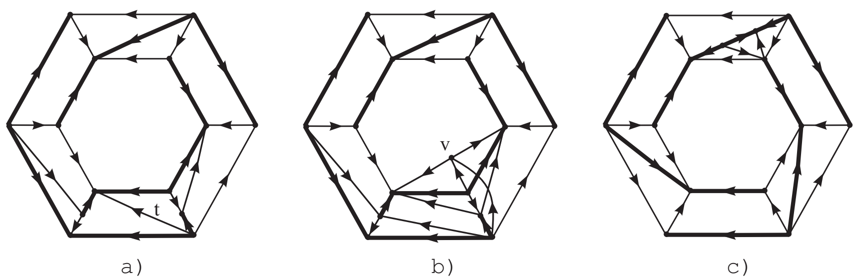

Fig. 3. Modifications of the previously known counterexamples.

Modification Two: The second modification requires disjoint node subsets R and S, which have directed paths from every r in R to every s in S, but no directed paths between distinct nodes within R or S. We add node v, and weight zero arcs rv and vs, for all r in R and s in S. The result is a counterexample since directed cuts have only changed by weight zero arcs. Furthermore, if $|R|$ and $|S|$ are at least two, then the result could be minimal. In (D_2, ω_2) and (D_3, ω_3), there are two R and S pairs with these properties. Figure 3 b) shows a minimal counterexample which is a modification of (D_3, ω_3).

Modification Three: This modification will be discussed in the full paper. In the meantime, readers can verify that the new counterexample, obtained from (D_1, ω_1), in Figure 3 c) still has the balanced trace $\{000, 011, 101, 110\}$.

4 An NP-Completeness Result

In the full paper, we will consider the following problem: Given a directed graph and a subset Y of its directed cuts, find a maximum sized set of disjoint Y-covers. We show that, unless P=NP, there is no algorithm that solves this problem in polynomial time, with respect to $|D| + |Y|$. The proof reduces the NP-complete problem NAE-SAT to deciding if two disjoint Y-covers exist.

5 Minimal Counterexamples

In the full paper, we strengthen our notion of minimality, and then derive properties of minimal counterexamples. When τ is two, these properties force the non-zero weight arcs to partition uniquely into three or more maximal s-paths. When restricted to three s-paths, we discover that (D_1, ω_1), $(D_2, \omega_2)^+$, and $(D_3, \omega_3)^+$ give the only possibilities. Furthermore, we show that no minimal counterexample can exist with four s-paths. Hence, the "smallest" minimal counterexamples have all been found.

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