ASSIGNMENT 3
DUE NOVEMBER 8TH, 2008

1. To express graph problems in predicate logic we first need to represent the graph. One approach is to let the domain be the set of vertices \( V \) and to create a relation \( edge \subset V^2 \), where \((x, y) \in edge\) iff the graph contains an edge between \( x \) and \( y \). Then we use a predicate letter, like \( q(x, y) \), to represent the \( edge \) relation. For example, \( \exists x \forall y \ q(x, y) \) is true iff the graph contains a vertex that is adjacent to all other vertices.

Given this representation of a graph, a subset of vertices \( \subset V \) can be represented by a 1-ary relation \( set \subset V \), where \( x \in set \) iff \( x \in V \). Then we use a predicate letter, like \( s(x) \), to represent the \( set \) relation. For example, \( \exists x \ s(x) \) is true iff \( V \) is a non-empty subset of vertices. Using these conventions, answer the following.

Note: Within your solutions you may want to use the predicate \( r(x, y) \) which is true iff \( x = y \).

a) Give a predicate formula that is true iff \( V \) is an independent set in the graph. [5 marks]

b) Give a predicate formula that is true iff \( V \) is a \textit{maximal} independent set in the graph. [5 marks]

2. For each of the five formulas below, provide an interpretation that is a model for that formula. Within your interpretations you must only use the relations from the following list.

- \( = \), where \((x, y) \in = \) iff \( x = y \)
- \( \neq \), where \((x, y) \in \neq \) iff \( x \neq y \)
- \( \leq \), where \((x, y) \in \leq \) iff \( x \leq y \)
- \( \geq \), where \((x, y) \in \geq \) iff \( x \geq y \)
- \( > \), where \((x, y) \in > \) iff \( x > y \)
- \( < \), where \((x, y) \in < \) iff \( x < y \)
- \( \bot \), where \((x, y) \in \bot \) iff \( x \bot y \). We write \( x \mid y \) if \( x \) \textit{divides} \( y \). In other words, \( x \mid y \) if there exists an integer \( k \) such that \( y = kx \). For example, \( 2 \mid 14 \) since \( 14 = 7 \cdot 2 \).
- \( \perp \), where \((x, y) \in \perp \) iff \( x \perp y \). We write \( x \perp y \) if \( x \) is \textit{co-prime} to \( y \). In other words, \( x \perp y \) unless there exists an integer \( k \) where \( k \mid x \) and \( k \mid y \) and \( k > 1 \). For example, \( 6 \perp 25 \). On the other hand, \( 4 \) is not co-prime with \( 14 \) since \( 2 \mid 14 \). [5 marks]

Furthermore, you may only use each relationship exactly \textit{once}. For example, if you use \( = \) for the model of the first formula then you may not use \( = \) for any of the other models. Likewise, if you use \( = \) for \( p_1 \) in the model of the last formula, then you may not use \( = \) for \( p_2 \) or \( p_3 \) in that model. Within each model you can use any one of the following three domains: \( \mathbb{Z} = \ldots, -1, 0, 1, \ldots \), \( \mathbb{Z}^+ = \{1, 2, 3, \ldots \} \), or \( \mathbb{Z}^- = \{\ldots, -3, -2, -1\} \).

a) \( \forall x \ p_1(a_1, x) \) [2 marks]

b) \( \forall x \ (p_1(x, a_1) \rightarrow p_2(x, a_1)) \) [2 marks]

c) \( \forall x \forall y \ (p_1(x, y) \rightarrow p_1(y, x)) \) [2 marks]

d) \( \forall x \forall y \exists z \ (p_1(x, y) \land p_1(y, z) \rightarrow p_1(x, z)) \) [2 marks]

e) \( \forall x \forall y \forall z \ (p_1(x, y) \land p_2(y, x) \rightarrow p_3(x, y)) \) [2 marks]

3. Prove that \( A = \forall x \ (\forall y \ (q(x) \rightarrow p(y)) \land \exists z \ (p(z) \land \neg q(x))) \) is logically equivalent to \( B = \neg \exists x \ \exists y \ (p(x) \oplus q(y)) \).

The proof must be of the form \( A \equiv \ldots \equiv B \) where each formula logically follows from the previous. [10 marks]

4. Use semantic tableau to show the following two results.

a) \( \forall x \ p(x) \rightarrow \exists x \ p(x) \) is satisfiable. [5 marks]

b) \( \forall x \ p(x) \rightarrow \neg \exists x \neg p(x) \) is unfalsifiable. [5 marks]

5. Express the predicate formula \( \forall x \exists y \ p(x, y) \) as a propositional formula, assuming that the domain is \( \{a, b, c\} \).

Let proposition \( p_{x,y} = T \) iff \((x, y)\) is in the relation represented by predicate \( p(x, y) \). [5 marks]

6. It is Hallowee’en and you are trapped in the magic castle of Hodjia, which is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet two inhabitants: Alice and Bob. Alice tells you that “of Bob and I, exactly one of us is a knight”. Bob says “only a knave would say that Alice is a knave!” So who is a knight and who is a knave? [5 marks]