1. a) $y$ represents the position of the leftmost 0.
   b) The following answers are all correct:
   - $x$ represents the position of the leftmost 1 where $x > y$
   - $x$ represents the position of the leftmost 1 that appears to the right of some 0
   - $x$ represents the position of the 1 in the leftmost 01.
   c) The following answers are all correct:
   - The values of $b$ are generated in reverse cool-lex order
   - The $i$th value of $b$ is the $(s^i) - i + 1$st value of $b$ in cool-lex order
   - The values of $b$ are generated by the following rule: Rotate the shortest prefix of $b$ that ends in 010 or 011 by one position to the right. If there is no prefix of $b$ that ends in 010 or 011 then rotate the entire value by one position to the right.
   d) $y := b[1] + y + 1$ is rewritten below (on the left).
   e) $x := x + 1 - (x - 1) * b[2] * (1 - b[1])$ is rewritten below (on the right).

$$
\begin{align*}
\text{if } b[1] &= 0 \\
&\text{ then } y := 1 \\
\text{ else } &\text{ end if}
\end{align*}
$$

$$
\begin{align*}
\text{if } b[1] &= 0 \land b[2] = 1 \\
&\text{ then } x := 2 \\
\text{ else } &\text{ end if}
\end{align*}
$$

2. a) $x = a + y$ is a loop invariant that would allow you to prove $\models \{ x \geq 0 \} \text{Copy}(x, y) \{ x = y \}$.
   b) $z = x \cdot a$ is a loop invariant that would allow you to prove $\models \{ y \geq 0 \} \text{Multiply}(x, y) \{ z = x \cdot y \}$.
   For the sake of reference, the actual proofs are included below.

**Copy**($x$, $y$)

1: $\{ x \geq 0 \}$
2: $\{ \text{True} \}$
3: $\{ x = x + 0 \}$
4: $a := x$
5: $\{ x = a + 0 \}$
6: $y := 0$
7: $\{ x = a + y \}$
8: while $a <> 0$ do
9: $\{ (x = a + y) \land (a \neq 0) \}$
10: $\{ x = a + y \}$
11: $\{ x = a - 1 + y + 1 \}$
12: $y := y + 1$
13: $\{ x = a - 1 + y \}$
14: $a := a - 1$
15: $\{ x = a + y \}$
16: end while
17: $\{ (x = a + y) \land (\neg a \neq 0) \}$
18: $\{ (x = a + y) \land (a = 0) \}$
19: $\{ x = y \}$

**Multiply**($x$, $y$)

1: $\{ y \geq 0 \}$
2: $\{ \text{True} \}$
3: $\{ 0 = x \cdot 0 \}$
4: $a := 0$
5: $\{ 0 = x \cdot a \}$
6: $z := 0$
7: $\{ z = x \cdot a \}$
8: while $a <> y$ do
9: $\{ (z = x \cdot a) \land (a \neq y) \}$
10: $\{ z = x \cdot a \}$
11: $\{ z + x = x \cdot a + x \}$
12: $\{ z + x = x \cdot (a + 1) \}$
13: $z := z + x$
14: $\{ z = x \cdot (a + 1) \}$
15: $a := a + 1$
16: $\{ z = x \cdot a \}$
17: end while
18: $\{ (z = x \cdot a) \land (\neg a \neq y) \}$
19: $\{ (z = x \cdot a) \land (a = y) \}$
20: $\{ z = x \cdot y \}$
3. a) Partial correctness for precondition \( \{ x > 0 \} \) and postcondition \( \{ z = x \mod 2 \} \) appears below. The loop invariant chosen is \( (y \mod 2 = x \mod 2) \land (y \geq -1) \) (there are equivalent statements for the invariant).

1: \( \{ x > 0 \} \)
2: \( \{ x \geq -1 \} \)
3: \( \{ (x \mod 2 = x \mod 2) \land (x \geq -1) \} \)
4: \( y := x \)
5: \( \{ (y \mod 2 = x \mod 2) \land (y \geq -1) \} \)
6: while \( y > 0 \) do
7: \( \{ (y \mod 2 = x \mod 2) \land (y \geq -1) \land (y > 0) \} \)
8: \( \{ (y \mod 2 = x \mod 2) \land (y \geq 1) \} \)
9: \( \{ (y - 2 \mod 2 = x \mod 2) \land (y - 2 \geq -1) \} \)
10: \( y := y - 2 \)
11: \( \{ (y \mod 2 = x \mod 2) \land (y \geq -1) \} \)
12: end while
13: \( \{ (y \mod 2 = x \mod 2) \land (y \geq -1) \land (y > 0) \} \)
14: \( \{ (y \mod 2 = x \mod 2) \land (y \geq -1) \land (y \leq 0) \} \)
15: \( \{ (y \mod 2 = x \mod 2) \land (-1 \leq y \leq 0) \} \)
16: \( \{ (y = 0 \rightarrow x \mod 2 = 0) \land (y \neq 0 \rightarrow x \mod 2 = 1) \} \)
17: if \( y = 0 \) then
18: \( \{ 0 = x \mod 2 \} \)
19: \( z := 0 \)
20: \{ \{ z = x \mod 2 \} \)
21: else
22: \( \{ 1 = x \mod 2 \} \)
23: \( z := 1 \)
24: \{ \{ z = x \mod 2 \} \)
25: end if
26: \( \{ z = x \mod 2 \} \)
27: \b) \( x \geq -1 \) is the weakest precondition that ensures total correctness because \( x \geq -1 \) is the weakest precondition that ensures partial correctness (see line 2) and the program terminates under this condition.

4. a) \[ wp(x := x \mod 2; y := x \mod 2, x < y) \]

   \[ = wp(x := x \mod 2, wp(y := x \mod 2, x < y)) \]

   \[ = wp(x := x \mod 2, x < x \mod 2) \]

   \[ = x + y < (x + y) \mod 2 \]

   \[ = (x + y < 0 \land y < 1) \lor (x + y > 0 \land y > 1) \]

Note: It is not correct to conclude \( y = 1 \) by dividing both sides of the second-last equation by \( x + y \), but no marks were taken off for doing this.

b) \[ wp(if \, x = 0 \, then \, y := y - 1 \, else \, y := 0 \, end, \, x = y) \]

   \[ = (x = 0 \land wp(y := y - 1, x = y)) \lor (x \neq 0 \land wp(y := 0, x = y)) \]

   \[ = (x = 0 \land wp(y := y - 1, x = y)) \lor (x \neq 0 \land x = 0) \]

   \[ = (x = 0 \land wp(y := y - 1, x = y)) \lor \text{False} \]

   \[ = x = 0 \land wp(y := y - 1, x = y) \]

   \[ = x = 0 \land x = y - 1 \]

   \[ = x = 0 \land y = 1 \]

Alternatively, the other rule for the if statement can be used

\[ wp(if \, x = 0 \, then \, y := y - 1 \, else \, y := 0 \, end, \, x = y) \]

\[ = (x = 0 \rightarrow wp(y := y - 1, x = y)) \land (x \neq 0 \rightarrow wp(y := 0, x = y)) \]

\[ = (x = 0 \rightarrow wp(y := y - 1, x = y)) \land (x \neq 0 \rightarrow x = 0) \]

\[ = wp(y := y - 1, x = y) \land x = 0 \]

\[ = x = y - 1 \land x = 0 \]

\[ = x = 0 \land y = 1. \]