Packing Dicycle Covers in Planar Graphs with No $K_5 - e$ Minor

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Abstract. We prove that the minimum weight of a dicycle is equal to the maximum number of disjoint dicycle covers, for every weighted digraph whose underlying graph is planar and does not have $K_5 - e$ as a minor ($K_5 - e$ is the complete graph on five vertices, minus one edge). Equality was previously known when forbidding K_4 as a minor, while an infinite number of weighted digraphs show that planarity does not guarantee equality. The result also improves upon results known for Woodall's Conjecture and the Edmonds-Giles Conjecture for packing dijoins. Our proof uses Wagner's characterization of planar 3-connected graphs that do not have $K_5 - e$ as a minor.

1 Introduction

Min-max theorems are fundamental to directed graph theory. For example, Menger's Theorem [7] proves that the minimum number of arcs separating node s from node t equals the maximum number of arc-disjoint dipaths from s to t. Reversing the roles of these objects gives another min-max theorem: the minimum number of arcs in a dipath from s to t equals the maximum number of arc-disjoint cuts separating s from t. Similarly, the celebrated Lucchesi-Younger Theorem [6] proves that the minimum number of arcs in a dijoin equals the maximum number of arc-disjoint dicuts. In all three cases, the min-max theorems can be extended from digraphs to weighted digraphs.

Still, many important min-max questions remain open or are untrue. Woodall's Conjecture [13] reverses the roles of the Lucchesi-Younger Theorem and asks if the minimum number of arcs in a dicut equals to the maximum number of arc-disjoint dijoins. Although Woodall's Conjecture remains one of the biggest open conjectures in graph theory, its weighted version (the Edmonds-Giles Conjecture [2]) is not true [9], [5], [12], [1]. In particular, Figure 1 shows that the Edmonds-Giles Conjecture is not true for planar digraphs. On the other hand, the conjecture was verified for series-parallel digraphs [8] (see also [10], [3], [4] which proves the conjecture for source-sink connected digraphs). In this

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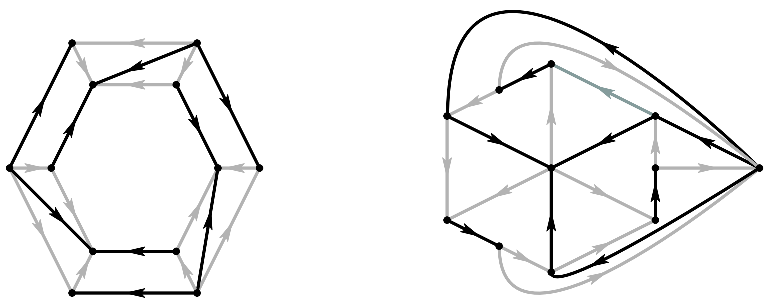

Fig. 1. A counterexample to the Edmonds-Giles Conjecture (left), and its planar dual (right). Light arcs have weight zero and dark arcs have weight one.

paper we narrow the gap between these two results by working on the planar dual problem. Along the way we introduces new techniques and lemmas that hold promise for future results in this challenging and important area.

Claim. If digraph D is planar and has no $K_5 - e$ minor, then for any arc weights, the minimum weight of a dicycle is equal to the maximum number of disjoint dicycle covers.

Our proof relies upon Wagner's characterization of 3-connected graphs that have no $K_5 - e$ minor. In particular, we show that dicycle covers can be glued across vertex cuts of size 1 and 2. Then, despite the global nature of dicycle covers, we are able to reduce the problem of finding dicycle covers locally. We redistribute weight around individual nodes and then eliminate arcs with zero weight or large weight. Furthermore, we employ a wye-delta reduction which removes a vertex of degree 3 and replaces it with edges between the vertex's neighbours.

Theorem 1 (Wagner). *If planar digraph D is 3-connected and has no $K_5 - e$ minor, then D is either a small complete graph, the envelope graph, or a wheel [11].*

2 Definitions, Notation, and Terminology

In this section we group together definitions, notation, and terminology necessary for the remainder of the paper. Graph-theoretic concepts that are open to different interpretations will be formally defined, while more standardized concepts will not. Included in this section are ideas that are common to many packing and covering theorems, so experienced readers may wish to skim this portion of the text. At the end of the section we introduce the notion of pushing weight into a cut, and point out its use in Remarks 1 and 2.

A *graph* $G = (V, E)$ is a set of vertices V and a set of edges E, where each edge is an unordered distinct pair of vertices. A *digraph* $D = (N, A)$ is a set of nodes N and a set of arcs A, where each arc is an ordered distinct pair of

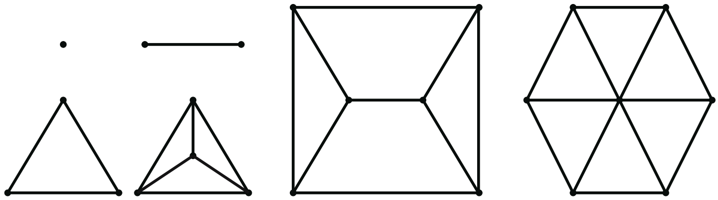

Fig. 2. From left to right: complete graphs K_1 through K_4, the envelope graph, and a wheel graph with seven vertices

nodes. Given a digraph $D = (N, A)$, its *underlying graph* is equal to (V, E), where $V = N$ and $E = \{xy : xy \in A \text{ or } yx \in A\}$. A *weighted digraph* (D, ω) is a digraph $D = (N, A)$ together with non-negative arc weights $\omega \in \{0, 1, 2, \ldots\}^A$.

Let C be a dicycle in D with arcs $A(C)$. The *weight* of C is denoted $\omega(C)$ and is equal to $\sum_{a \in A(C)} \omega(a)$. The minimum weight of a dicycle in (D, ω) is denoted $\tau(D, \omega)$. Let $J \subseteq A$ be a subset of arcs. J *covers* C if $J \cap A(C) \neq \emptyset$. J is a *(dicycle) cover* of D if J covers every dicycle in D. A cover is *minimal* if every proper subset of it is not a cover. A collection of arc subsets $J_1, J_2, \ldots, J_k \subseteq A$ are *disjoint* in (D, ω) if at most $\omega(a)$ of the covers use a, for all $a \in A$. The maximum number of disjoint covers in (D, ω) is denoted $\nu(D, \omega)$. Notice that $\nu(D, \omega) \leq \tau(D, \omega)$. If equality holds then we say that (D, ω) *packs*; otherwise (D, ω) does not pack. Finally, a collection of $\tau(D, \omega)$ disjoint covers is called a *packing* of covers. Central to finding a packing of covers is the pursuit of special covers which we will call valid and accommodating. Let $v_J \in \{0, 1\}^A$ be an incidence weighting for $J \subseteq A$, where $v_J(a) = 1$ if $a \in J$, and $v_J(a) = 0$ if $a \notin J$. We say that J is *valid* in (D, ω) if $\omega - v_J$ has only non-negative entries; that is, if $\omega(a) > 0$ for each $a \in J$. J *accommodates* dicycle C in (D, ω) if $\omega'(C) \geq \tau(D, \omega) - 1$, where $\omega' = \omega - v_J$. Furthermore, J is *accommodating* in (D, ω) if every dicycle in C is accommodated by J in (D, ω). Notice that if J is accommodating in (D, ω) then $\tau(D, \omega - v_J) = \tau(D, \omega) - 1$; that is, J leaves enough room for the possibility of finding $\tau(D, \omega) - 1$ disjoint covers after its *removal* forms $(D, \omega - v_J)$. Notice that every cover in a packing is valid and accommodating, and that the ability to always find a valid and accommodating cover allows one to construct a packing of covers.

Let $X \subseteq N$ be a set of nodes in digraph (N, A). We let $\overline{X} = N - X$, and the *cut induced by* X is represented by $\delta(X)$ and is equal to $\delta^{\text{in}}(X) \cup \delta^{\text{out}}(X)$, where $\delta^{\text{in}}(X) = \{xy \in A : x \in \overline{X} \text{ and } y \in X\}$ and $\delta^{\text{out}}(X) = \{xy \in A : x \in X \text{ and } y \in \overline{X}\}$. If $\delta^{\text{out}}(X) = \emptyset$ then we say that $\delta^{\text{in}}(X)$ is a *dicut*.

A *digon* is a dicycle of length 2, and any arc that is in a digon is called a *digon arc*.

Given a graph $G = (V, E)$ and $e \in E$, we let $G \backslash e$ represent the result of deleting edge e, and we let G/e represent the result of contracting edge e. Likewise, given $v \in V$, we let $G \backslash v$ be the result of deleting vertex v and every edge that is incident with v. For a subset of vertices $X \subseteq V$, we let $G[X]$ be the result of deleting every vertex in $\overline{X}$. In graph $G = (V, E)$, a *k-separation* is a pair of subgraphs (G_1, G_2) where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, such that

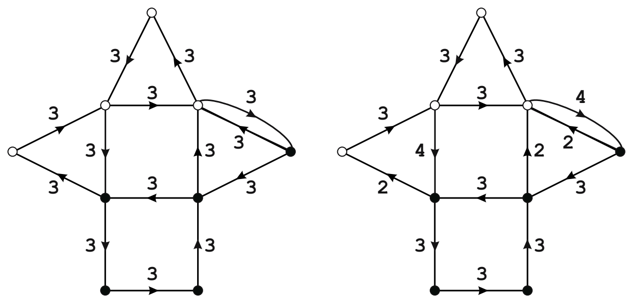

Fig. 3. Before and after pushing into $\delta(X)$, where X is given by the black nodes

$E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $V_1 \cup V_2 = V$, and $|V_1 \cap V_2| = k$. If a graph does not have an i-separation for any $i \leq k-1$, then we say that it is *k-connected.* We use analogous notation for digraphs and weighted digraphs, except that for weighted digraphs we implicitly assume that contraction and deletion will result in weights that are restricted to the remaining arcs.

Given a weighted digraph (D, ω) with $D = (N, A)$, and $X \subset N$, then *pushing into* $\delta(X)$ results in a new weighting for D, denoted by $\rho(\omega, X) = \omega'$ where

$$\omega'(a) = \begin{cases} \omega(a) + 1 & \text{if } a \in \delta^{\text{in}}(X) \\ \omega(a) - 1 & \text{if } a \in \delta^{\text{out}}(X) \\ \quad \omega(a) & \text{otherwise} \end{cases}$$

Remark 1. If C is a dicycle in (D, ω), then $\omega(C) = \omega'(C)$ where $\omega' = \rho(\omega, X)$, for $X \subseteq N$. In particular, $\tau(D, \omega) = \tau(D, \omega')$.

Remark 2. J is accommodating in $(D, \omega) \iff J$ is accommodating in $(D, \rho(\omega, X))$, for $X \subseteq N$.

Remarks 1 and 2 follow from the fact that $|A(C) \cap \delta^{\text{in}}(X)| = |A(C) \cap \delta^{\text{out}}(X)|$ for any dicycle C, and any subset of nodes X. It is worth noting that Remarks 1 and 2 hold regardless of how many cuts we push into, and whether or not we push into the same cut more than once. To perform successive pushes, let us define

$$\rho^0(\omega, X) = \omega$$
$$\rho^i(\omega, X) = \rho(\rho^{i-1}(\omega, X), X)$$

Often we want to push as much weight into a cut as possible, and we also want to avoid making an arc have negative weight, so we are constrained by the minimum weight of an outgoing arc in the cut. For this reason we introduce the following notation: let $\rho^*(\omega, X)$ be shorthand for $\rho^i(\omega, X)$ where

$$i = \min\{\omega(a) : a \in \delta^{\text{out}}(X)\}$$

and $i = \tau(D, \omega)$ if $\delta^{\text{out}}(X) = \emptyset$. To aid in the readability of this document, we suggest that $\rho(\omega, \overline{X})$ be verbalized as "pushing out of $\delta(X)$" as opposed to "pushing into $\delta(\overline{X})$". Finally, we generally wish to push into or out of cuts surrounding a single node, and so we will use the notation $\rho(\omega, x)$ as a short-form for $\rho(\omega, \{x\})$, and $\rho(\omega, \overline{x})$ as a short-form for $\rho(\omega, \overline{\{x\}})$, for node x.

3 Connectivity Lemmas

In this section we show that packings can be combined across dicuts, 1-separations, and 2-separations whose overlapping vertices form a digon. For this entire section we let (D, ω) be a weighted digraph with $D = (N, A)$.

Lemma 1 (Dicut). *Suppose that $\delta^{in}(S)$ is a dicut in D. Let $D_1 = D[S]$, let $D_2 = D[\overline{S}]$, and let ω_i be ω restricted to D_i for each $i = 1, 2$. If (D_1, ω_1) packs and (D_2, ω_2) packs, then (D, ω) packs.*

Proof. Note that $\tau(D_i, \omega_i) \geq \tau(D, \omega)$ for each $i = 1, 2$. Suppose that (D_i, ω_i) packs for each $i = 1, 2$. Then there exists a packing including $J^i_1, \ldots, J^i_{\tau(D,\omega)}$ in (D_i, ω_i) for each $i = 1, 2$. Let $J_j = J^1_j \cup J^2_j$, for $1 \leq j \leq \tau(D, \omega)$. Clearly J_j is a cover of D, for $1 \leq j \leq \tau(D, \omega)$. Thus, $J_1, \ldots, J_{\tau(D,\omega)}$ is a packing for (D, ω).

Lemma 2 (1-separation). *Let (D_1, D_2) be a 1-separation of D. Let ω_i be ω restricted to D_i for each $i = 1, 2$. If (D_1, ω_1) packs and (D_2, ω_2) packs, then (D, ω) packs.*

Proof. The proof is identical to the proof of Lemma 1.

Lemma 3 (2-separation). *Let (D'_1, D'_2) be a 2-separation of D such that D'_1 and D'_2 share vertices x and y. Let ω_i be ω restricted to D_i for each $i = 1, 2$. Let α_i be the minimum weight of a dipath from x to y in (D'_i, ω_i) for each $i = 1, 2$. Assume that $\alpha_1 \leq \alpha_2$, and let $\alpha = \min\{\tau(D, \omega), \alpha_1\}$. Let $e = xy, f = yx$ be new arcs. For each $i = 1, 2$, let $D_i = D'_i \cup \{e, f\}$, let $\omega_i(e) = \alpha$, and let $\omega_i(f) = \tau(D, \omega) - \alpha$. If (D_1, ω_1) packs and (D_2, ω_2) packs, then (D, ω) packs.*

Proof. Let $\tau = \tau(D, \omega)$. We claim that $\tau(D_i, \omega_i) = \tau$ for each $i = 1, 2$. We prove it for D_1; the proof is analogous for D_2. Suppose there exists a dicycle C in D'_1 such that $\omega_1(C) < \tau$. Clearly, C must contain e or f. If $f \in A(C)$ then $C - f$ gives a dipath from x to y in D_1 such that $\omega_1(C - f) < \alpha \leq \alpha_1$, which contradicts the choice of α_1. If $e \in A(C)$ then $\alpha_1 = \alpha < \tau$, and $C - e$ gives a path in D_1 such that $\omega(C - e) = \omega_1(C - e) < \omega - \alpha_1$. Let Q be a minimum length dipath from x to y in $(D'_1, \omega_1[D'_1])$, that is, $\omega(Q) = \alpha_1$. Then $\omega((C - e) \cup Q) < \tau - \alpha_1 + \alpha_1 = \tau$, and hence $(C - e) \cup Q$ contains a dicycle Z in D such that $\omega(Z) < \tau$, which is a contradiction. Hence, $\tau(D'_1, \omega_1) = \tau$.

If $\tau(D_i, \omega_i) = \nu(D_i, \omega_i)$ for each $i = 1, 2$, then there exists a packing including τ covers of D'_i, say $\{J^i_1, \ldots, J^i_\tau\}$, for each $i = 1, 2$.; We may assume that for each $i = 1, 2$, we have $e = xy \in J^i_j$, for $1 \leq j \leq \alpha$, and $f = yx \in J^i_j$, for $\alpha + 1 \leq j \leq \tau$. Let $J_j = (J^1_j \cup J^2_j) - \{e, f\}$, $1 \leq j \leq \tau$. We claim that each J_j is a cover of D, for $1 \leq j \leq \tau$. In fact, if C is a dicycle contained in D_1 or in D_2 then J_j clearly intersects C. Otherwise, x and y are vertices in C, and C can be partitioned in two paths P and Q from x to y and from y to x, respectively. Without loss of generality, say that P is contained within the arcs of D_1, and Q is contained within the arcs of D_2. We have two cases to consider.

(a) $1 \leq j \leq \alpha$: note that $P \cup f$ is a dicycle in D'_1 and recall that J^1_j is a cover of D'_1. Since $f \notin J^1_j$, then J^1_j (and hence, J_j) intersects P.
(b) $\alpha + 1 \leq j \leq \tau$: note that $Q \cup e$ is a dicycle of D'_2 and recall that J^2_j is a cover of D'_2. Since $e \notin J^2_j$, then J^2_j (and hence, J_j) intersects Q.

Thus, in both cases each J_j is a cover of D, and hence, $\{J_1, \ldots, J_\tau$ is a packing in (D, ω).

4 Contraction and Deletion Lemmas

In this section, we continue to show how packings in a smaller weighted digraph can be extended to packings in a larger original weighted digraph called (D, ω) with $D = (N, A)$. However, in this section there is a single smaller weighted digraph, and it arises not from dicuts or separations, but instead from individual arcs that are deleted or contracted. In particular, we associate deletion with arcs of weight at least $\tau(D, \omega)$, and contraction with non-transitive arcs of weight 0. We require non-transitive arcs for contracting since we do not wish to introduce new dicycles. On the other hand, we are not concerned with removing dicycles when deleting an arc of weight at least $\tau(D, \omega)$ since the arc can be added to every cover of the smaller weighted digraph.

We also extend these results by pushing weight into a cut $\delta(X)$ to bring an arbitrary arc to weight $\tau(D, \omega)$, or a non-transitive arc to weight 0. We point out that it is always possible to create arcs of weight 0 in a cut, however it is not always possible to create arcs of weight $\tau(D, \omega)$ in a cut. In particular, if we are pushing into $\delta(X)$ with $\delta^{\text{in}}(X) \neq \emptyset$, then it must be that $\max_{a \in \delta^{\text{in}}(X)} \omega(a) + \min_{a \in \delta^{\text{out}}(X)} \omega(a) \geq \tau(D, \omega)$. After manipulating weights and forming a smaller weighted digraph, we are not interested in revealing an entire packing for (D, ω), but merely a single valid accommodating cover. Because of Remarks 1 and 2, our challenge is ensuring that at least one cover in the smaller weighted digraph is valid in (D, ω). For this reason, $|\delta^{\text{out}}(X) \cap \{a \in A : \omega(a) = 0\}|$ becomes important. If the value is strictly less than $\tau(D, \omega)$, then we can ensure that at least one of the $\tau(D, \omega)$ covers found in the smaller weighted digraph will be valid in (D, ω); otherwise, we can think of $\delta(X)$ as being *protected* against such an argument.

Lemma 4 (Contract). *(a) Suppose $\exists$ non-transitive $a \in A$ with $\omega(a) = 0$. If $(D, \omega)/a$ packs, then (D, ω) packs.*
(b) Suppose $\exists$ non-transitive $a \in \delta^{out}(X)$ and $X \subseteq N$, such that $\omega(a) = \min\{\omega(b) : b \in \delta^{out}(X)\}$ and $|\delta^{in}(X) \cap \{b \in A : \omega(b) = 0\}| < \tau(D, \omega)$. If $(D, \rho^*(\omega, X))/a$ packs, then (D, ω) has a valid accommodating cover.

Proof. (a) Since a is non-transitive, it means that the dicycles in D and D/a are identical (except that some dicycles in D/a no longer include a). Therefore,

by Remark 1 and since $\omega(a) = 0$, we have that $\tau(D,\omega) = \tau(D,\omega)/a$, and that a cover in D/a is a cover in D. Therefore, the packing in $(D,\omega)/a$ is also a packing in (D,ω).
(b) Let $\omega' = \rho^*(\omega, X)$. Since a is non-transitive, it means that the dicycles in D and D/a are identical (except that some dicycles in D/a no longer include a). Therefore, by Remark 1 and since $\omega'(a) = 0$, we have that $\tau(D,\omega) = \tau(D,\omega') = \tau((D,\omega')/a)$, and that a cover in D/a is a cover in D. Let $J_1, \ldots, J_{\tau(D,\omega)}$ be a packing in (D,ω'). From Remark 2, each J_i is accommodating in (D,ω). Furthermore, since the only arcs that have $\omega(b) = 0$ and $\omega'(b) > 0$ are contained in $\delta^{\text{in}}(X)$, and since $|\delta^{\text{in}}(X) \cap \{b \in A : \omega(b) = 0\}| < \tau(D,\omega)$, it must be that one of the J_i covers is also valid in (D,ω).

Lemma 5 (Delete). *(a) Suppose $\exists a \in A$ with $\omega(a) \geq \tau(D,\omega)$. If $(D,\omega)\backslash a$ packs, then (D,ω) packs.*
(b) Suppose $\exists a \in A$ and $X \subseteq N$, such that $a \in \delta^{in}(X)$ and $\max_{b\in\delta^{in}(X)} \omega(b) + \min_{b\in\delta^{out}(X)} \omega(b) \geq \tau(D,\omega)$ and $|\delta^{in}(X) \cap \{b \in A : \omega(b) = 0\}| < \tau(D,\omega)$. If $(D,\rho^*(\omega,X))\backslash a$ packs, then (D,ω) has a valid accommodating cover.

Proof. (a) Deleting a does not decrease the minimum weight of a dicycle, so there is a packing of covers in $(D,\omega)\backslash a$ that includes $J_1, \ldots, J_{\tau(D,\omega)}$. Notice that each J_i covers every dicycle in D, except possibly for some dicycles containing a. Therefore, since $\omega(a) \geq \tau(D,\omega)$, we have that $J_1 \cup \{a\}, \ldots, J_{\tau(D,\omega)}$ is a packing for (D,ω).
(b) Let $\omega' = \rho^*(\omega, X)$. Notice that $\omega'(a) \geq \tau(D,\omega)$. By Remark 1, and since deleting a does not decrease the minimum weight of a dicycle, we have that $\tau(D,\omega) = \tau(D,\omega') \geq \tau((D,\omega')\backslash a)$. Therefore, there is a packing of covers in $(D,\omega)\backslash a$ that includes $J_1, \ldots, J_{\tau(D,\omega)}$. Notice that each J_i covers every dicycle in D, except possibly for some dicycles containing a. Therefore, $J_1 \cup \{a\}, \ldots, J_{\tau(D,\omega)}$ are all covers in (D,ω). From Remark 2, each J_i is accommodating in (D,ω). Furthermore, since the only arcs that have $\omega(a) = 0$ and $\omega'(a) > 0$ are contained in $\delta^{\text{in}}(X)$, and since $|\delta^{\text{in}}(X) \cap \{b \in A : \omega(b) = 0\}| < \tau(D,\omega)$, it must be that one of the J_i covers is also valid in (D,ω).

5 Proof of Claim

Now we are ready to use the results of the two previous sections in order to prove Claim 1, which is restated below as Claim 5.

Claim. (D,ω) packs whenever the underlying graph of D has no $K_5 - e$ minor.

To prove Claim 5, we show that a smallest counterexample cannot exist. We call such a counterexample (D_m, ω_m), where $D_m = (N_m, A_m)$. For notational convenience, let $\tau_m = \tau(D_m, \omega_m)$. We choose (D_m, ω_m) to be smallest in the sense that it first minimizes τ, then the number of nodes, and finally the number of arcs. Hence, (D,ω) packs whenever one of the following holds:

M1: $\tau(D, \omega) < \tau_m$

M2: $\tau(D, \omega) = \tau_m$ and $|N| < |N_m|$

M3: $\tau(D, \omega) = \tau_m$ and $|N| = |N_m|$ and $|A| < |A_m|$

Remark 3. D_m is 3-connected.

From these choices and the results in Section 3, we have proven that D_m is 3-connected (Remark 3). Therefore, by Theorem 1 it must be that the underlying graph of D_m is a small complete graph, the envelope graph, or a wheel. Without too much difficulty, we can eliminate the possibility of K_2 and K_3. Furthermore, a wheel with three vertices is K_3, and a wheel with four vertices is K_4. Therefore, we have the following:

Remark 4. The underlying graph of D_m is a member of the set $S = \{ K_4, E, W_5, W_6, W_7, \ldots\}$, where E is the envelope graph, and W_i is a wheel with i vertices.

Fortunately, each member of S contains vertices of degree 3. In fact, every vertex has degree 3 except for the middle vertex in the wheels. Even more fortuitously, if we perform a wye-delta reduction on any of the graphs in S, then we either get K_3, or a graph in S with one less vertex. (If v is a vertex of degree 3 with neighbours x, y, z, then a wye-delta reduction is the result of removing vertex v and adding edges xy, xz, yz.)

Remark 5. D_m has no dicuts.

Since D_m has no dicuts, there is at least one arc entering and leaving each node in D_m. Therefore, Figure 4 shows the possible configurations for the directed versions of its degree 3 vertices (without distinguishing between neighbours).

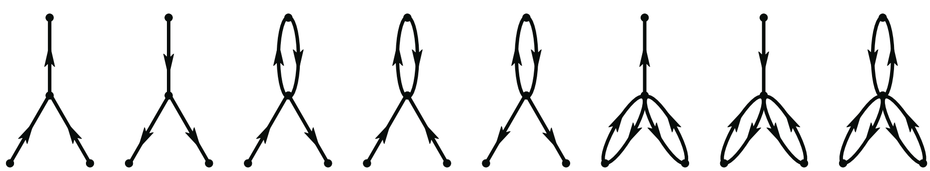

Fig. 4. From left to right: Configurations 1 through 8

By using the ideas from Section 4, let us now eliminate all but the first three configurations. From (M2) and Lemma 5, we cannot push any arc in A_m to weight τ_m, unless the cut we are pushing on is protected by $|\delta^{\text{out}}(X) \cap \{a \in A : \omega_m(a) = 0\}| \geq \tau_m$. Notice that pushing a digon arc to weight zero is equivalent to pushing its partnered digon arc to weight τ_m. Therefore, Configurations 6, 7, 8 cannot appear in (D_m, ω_m). For Configuration 4 and 5, notice that in both cases there is a digon arc a that would appear in no other dicycle except with the digon arc that it is partnered with, and we will call b. Therefore, we can delete a, find τ_m covers by minimality, and then extend these to covers of (D_m, ω_m) simply by adding a to the covers that b is not included in. Hence, Configurations 4 and 5 cannot appear in (D_m, ω_m).

Configuration 3 is slightly more difficult to eliminate. Let us label the nodes and arc weights of Configuration 3 as in the left portion of Figure 5. As in the previous paragraph, we cannot push a digon arc to weight 0. Therefore, we have the following conditions:

$$s < r$$
$$t < q$$

Our strategy is now to replace this configuration by using a wye-delta reduction which is illustrated in the right portion of Figure 5. We will call the newly formed weighted digraph (D, ω). In particular, we construct (D, ω) so that every dicycle in (D_m, ω_m) has a corresponding dicycle in (D, ω) with the same weight, except for the digon, which is not transferred to (D, ω). From the previous discussion on wye-delta reductions on the set of graphs S, and from (M2), (D, ω) packs. From the arc weights shown in Figure 5, it is clear that $\tau(D, \omega) = \tau_m$, and so we have a packing $J_1, \ldots, J_{\tau_m}$ for (D, ω). We now wish to show that this packing gives a valid accommodating cover for (D_m, ω_m). Before proceeding, we must constrain which sets of arcs the individual covers can contain by introducing Lemma 6.

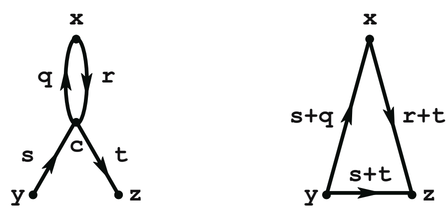

Fig. 5. Delta-wye reduction for Configuration 3

Lemma 6. *Let C be a cycle that is not a dicycle, and of the two directions of arcs within C, let one direction be called* forward *and the other* backward. *If J is a minimal cover and includes every forward arc of C, then J includes at least one backward arc of C.*

Proof. Otherwise, since J is minimal, for every forward arc a in C, there is a dicycle C_a such that $A(C_a) \cap J = \{a\}$. However, if J does not include any backward arc of C, then we contradict that J is a cover, since taking C_a for each forward arc a in C, together with the backward arcs of C, gives a dicycle that is not covered by J.

From Lemma 6, we know that the possible sets of arcs contained in each J_i are given by Figure 6. If J_i is of the first type, then it can be converted into a

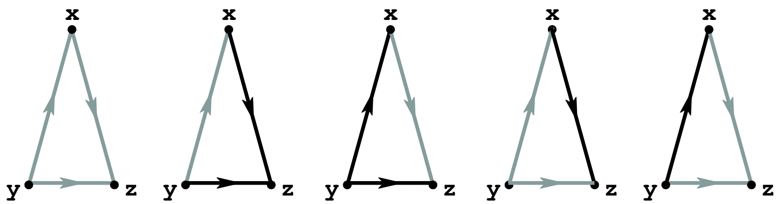

Fig. 6. From left to right, the five possible arc sets included in each J_i are given by the light arcs

valid accommodating cover for (D_m, ω_m) by replacing arcs yx, yz, xz by cx, cz (Figure 7 (left)). If J_i is of the second type, then it can be converted into a valid accommodating cover for (D_m, ω_m) by replacing arc yx with cx (Figure 7 (center)). If J_i is of the third type, then it can be converted into a valid accommodating cover for (D_m, ω_m) by replacing arc xz with xc (Figure 7 (right)). Therefore, we can eliminate Configuration 3 by guaranteeing that at least one of $J_1, \ldots, J_\tau$ is of one of the first three types given by Figure 6. Fortunately, the last two types given by Figure 6 can be used at most $s+t$ times by the weight given in Figure 5. From the above discussion on pushing, $s < r$ and $t < q$, so $s+t < r+q$. However, $r+q$ are the weights of a digon, so $s+t < \tau_m$, and therefore, at least one of $J_1, \ldots, J_{\tau_m}$ can be converted into a valid accommodating cover for (D_m, ω_m).

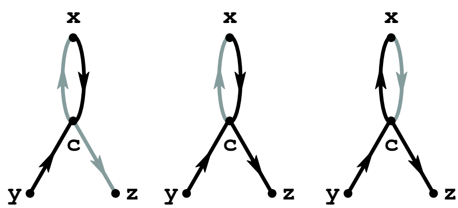

Fig. 7. From left to right, converting the first three possible arc sets into valid accommodating covers for (D_m, ω_m)

Therefore, we have now shown that every degree 3 vertex in the underlying graph of D_m is of Configuration 1 or Configuration 2 in Figure 4. Furthermore, from the results of Section 4, we know that at least one of the cuts surrounding these degree 3 vertices must be protected by $|\delta^{\text{out}}(X) \cap \{a \in A : \omega_m(a) = 0\}| \geq \tau_m$. Therefore, we must have that $\tau_m = 2$ and the arc weights must be as shown in Figure 8.

The endgame for our proof now consists of showing that when the underlying graph of (D_m, ω_m) is the envelope graph, or a wheel, and $\tau_m = 2$, and the degree 3 nodes of (D_m, ω_m) are given by Figure 8, then (D_m, ω_m) packs. It is not difficult to verify by hand that the envelope graph cannot possibly have nodes of degree 3 that are consistent with Figure 8. Therefore, we turn our attention to the infinite class of wheels.

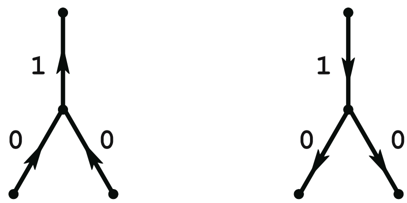

Fig. 8. Every degree 3 node in (D_m, ω_m) must be isomorphic to one of the above

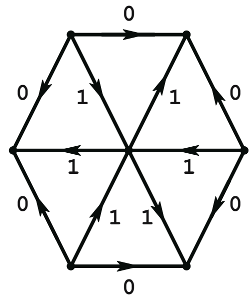

Fig. 9. If (D_m, ω_m) is a wheel, then it must have an odd number of nodes as above

Fortunately, the heavy restrictions force the wheels to have an odd number of vertices, with the outer cycle alternating in direction with weight zero arcs, and the inner arcs alternating in direction around the wheel with weight one. Figure 9 shows an example with seven nodes. Given such a weighted digraph, it is easy to find a packing of two covers, since we can let $J_1 = \{xc : xc \in A_m\}$ and $J_2 = \{cx : cx \in A_m\}$, where c is the center node in the wheel.

Therefore, we have show that (D_m, ω_m) does in fact pack. Therefore, there is no minimal counterexample, and we have verified our original claim. We conclude the paper by pointing out the corollary that it has for the Edmonds-Giles Conjecture, and less generally, Woodall's Conjecture.

Corollary 1. *Let (D, ω) be a weighted digraph, and let P be the planar dual of the $K_5 - e$ graph. If the underlying graph for D does not contain P as a minor, then the maximum number of disjoint dijoins is equal to the minimum weight of a dicut.*

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