Instructions: Students in MATH 4805 / COMP 4805 answer seven questions among (1)-(8), and students in MATH 5605 answer eight questions among (1)-(9).

Figure 1. Error message from Carleton’s Faculty/Staff Computing Account Password Reset Form https://luminis.carleton.ca/secure-cgi/passwd.cgi

(1) (10 marks) You do not need to justify your answers to the following questions.
   (a) (5 marks) Give a finite automaton that accepts $L_s$ and a regular expression that generates $L_s$ for $L_s = \{ w \in \{0,1\}^* \mid w$ has suffix 010$\}$.
   (b) (5 marks) Give a finite automaton that accepts $L_o$ and a regular expression that generates $L_o$ for $L_o = \{ w \in \{0,1\}^* \mid w$ has an even # of 0s and an odd # of 1s$\}$.

(2) (10 marks) To obtain a CONNECT account at Carleton University, members of the faculty and staff must choose a connect password that satisfies a number of basic rules. The bulleted list in Figure 1 states a number of the rules explicitly, and the error message at the top of the figure shows that additional rules need to be satisfied.
   (a) (2 marks) Why are the connect passwords a recognizable language?
Consider an alternate definition of a password based on the connect password. Our alphabet will be $A = L \cup U \cup D \cup P \cup Q$ where $L = \{a, b, \ldots, z\}$ (lowercase letters), $U = \{A, B, \ldots, Z\}$ (uppercase letters), $D = \{0, 1, \ldots, 9\}$ (digits), $P = \{!, ^, ., ?\}$ (valid punctuation) and $Q = \{&, @, #, \{, \}\}$ (invalid punctuation). A password is any string $w \in A^*$ that satisfies all of the following conditions

- $|w| \geq 6$
- must not have a substring of length three that is also a substring of length three in your login name
- must not have a substring of length three of the form $aaa$ for $a \in A$
- must not have a substring that is a postal code
- the reverse of $w$ and the Kleene closure of $w$ satisfy the above substring conditions
- must not contain any symbol in $Q$
- must contain at least four unique symbols
- the first six symbols in $w$ must contain at least one symbol from $L \cup U$, one symbol from $D$, and one symbol from $P$.

In this description assume that your login name is your first name, a postal code is any string in $U \times D \times U \times D \times U \times D$, and a license plate is any string in $U \times U \times U \times D \times D \times D$.

(b) (8 marks) Justify that $L = \{w \in A^* \mid w$ is a password}\} is recognizable. In your answer you must use the closure properties of recognizable languages. Apply these closure properties using at least one finite automata, at least one non-deterministic or $\epsilon$ finite automata, and at least one regular expression. Do not justify the correctness of the individual automata and regular expressions.

(3) (10 marks) Prove your answers to the following questions.

(a) (5 marks) Is $L_p = \{w \in \{0, 1\}^* \mid w$ is a palindrome\} recognizable?

(b) (5 marks) Is $L_e = \{w \in \{0, 1\}^* \mid w$ has an equal # of 01 and 10 substrings\} recognizable?

(4) (10 marks) Recall that a language $L \subseteq A^*$ satisfies the Pumping Property if the following holds: $\exists$ number $n$ such that $\forall w \in L$ with $|w| \geq n$ $\exists x, y, z \in A^*$ such that

- $w = xyz$,
- $|xy| \leq n$,
- $|y| > 0$, and
- $\forall i \geq 0 \\text{xy}^i z \in L$.

(a) (3 marks) Prove that the following language $L_a$ satisfies the Pumping Property

$L_a = \{w \in \{0, 1, 2\}^* \mid w = 0^a1^b2^b$ or $w = 1^a2^b$ for some $a, b \geq 0\}$.

A language $L \subseteq A^*$ satisfies the Generalized Pumping Property if the following holds: $\exists$ number $n$ such that $\forall wu \in L$ with $|w| \geq n$ $\exists x, y, z \in A^*$ such that

- $w = xyz$,
- $|xy| \leq n$,
- $|y| > 0$, and
- $\forall i \geq 0 \text{pxy}^i z \in L$.
(b) (1 mark) Informally, what is the difference between the Pumping Property and
the Generalized Pumping Property?
(c) (3 marks) Prove that $L_a$ does not satisfy the Generalized Pumping Property.
(d) (3 marks) Prove that the Generalized Pumping Property holds for any recognizable
language.

(5) (10 marks) Answer two of the following three questions.
(a) (5 marks) Prove or disprove the following. If $L$ is a recognizable language over
alphabet $A$, then so is $L/s$ for any $s \in A^*$. The operation $/$ is known as a
quotient and is defined as $L/s = \{w \in L \mid w = ps \text{ for some } p \in A^*\}$.
(b) (5 marks) [COMP 4805] Suppose $L_1$ and $L_2$ are recognizable languages such that
$L_1 = L(M_1)$ and $L_2 = L(M_2)$ for finite automata $M_1 = (S_1, A, i_1, \delta_1, F_1)$ and
$M_2 = (S_2, A, i_2, \delta_2, F_2)$. Provide an $\epsilon$-finite automata $M = (S, A, I, \delta, F)$ such
that $L(M) = L_1 \otimes L_2 = \{w \in A^* \mid w \text{ is in exactly one of } L_1 \text{ and } L_2\}$ where $\otimes$
 denotes xor.
(c) (5 marks) [MATH 4805/5605] If $A$ and $B$ are two alphabets, then $f : A^* \to B^*$
is a monoid homomorphism if $f(\epsilon) = \epsilon$ and $f(uv) = f(u)f(v)$ for all $u, v \in A^*$. If
$L$ is a recognizable language and $f$ is a monoid homomorphism, then prove
that $f(L) = \{f(w) \mid w \in L\}$ is recognizable.

(6) (10 marks) Consider the following finite automaton $M = (S, A, i, \delta, F)$.

(a) (2 marks) Prove that $s_1 \not\sim_M s_2$. In other words, provide any string $w$ such that
either (i) $\delta^*(s_1, w) \in F$ and $\delta^*(s_2, w) \notin F$, or (ii) $\delta^*(s_1, w) \notin F$ and $\delta^*(s_2, w) \in F$.
(b) (5 marks) Compute the partition table for $S/\sim_M$. Show the auxiliary tables
that distinguish between states for strings of length $0, 1, 2, \ldots$.
(c) (3 marks) Draw the minimal finite automaton. What language does it accept?

(7) (10 marks) Reduce the NFA given by the following transition to a deterministic FA
by using the accessible subset construction.

In the above table the final states have $*$ next to them. Show your work.
(8) (10 marks) Consider the following ε-finite automaton \( M = (S, A, i, \delta, F) \).

\[
\begin{array}{c}
\text{state } x & \text{\(E(x)\)} & \text{\(E(x) \cdot a\)} & \text{\(E(x) \cdot b\)} & \text{\(E(E(x) \cdot a)\)} & \text{\(E(E(x) \cdot b)\)} \\
\hline
s_1 &  &  &  &  &  \\
s_2 &  &  &  &  &  \\
s_3 &  &  &  &  &  \\
s_4 &  &  &  &  &  \\
\end{array}
\]

(a) (2 marks) Determine the \(\epsilon\)-closure \(E(x)\) for each \(x \in S\).
(b) (2 marks) Determine \(E(x) \cdot a\) (i.e., \(\delta(E(x), a)\)) for each \(x \in S\) and \(a \in A\).
(c) (2 marks) Determine \(E(E(x) \cdot a)\) for \(x \in S\) and \(a \in A\).
(d) (4 marks) Remove the \(\epsilon\)-transitions to provide an NFA that accepts \(L(M)\).

(9) (10 marks) [MATH 5605 only] Prove or disprove that recognizable languages are closed under shuffles. Given two languages \(L_1\) and \(L_2\) over alphabet \(A\), the shuffle of \(L_1\) and \(L_2\) is \(\{a_1b_1a_2b_2\cdots a_kb_k \mid a_1a_2\cdots a_k \in L_1\) and \(b_1b_2\cdots b_k \in L_2\}\).