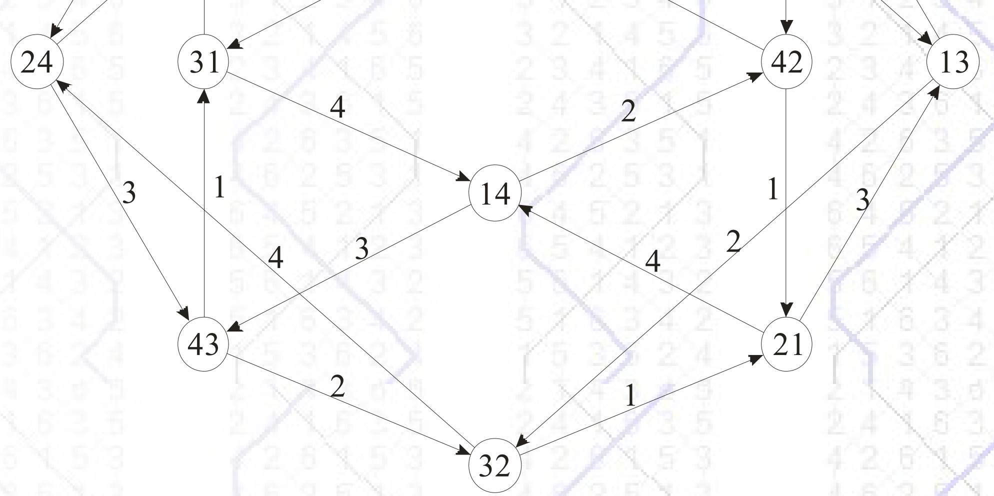


Universal Cycles for Permutations

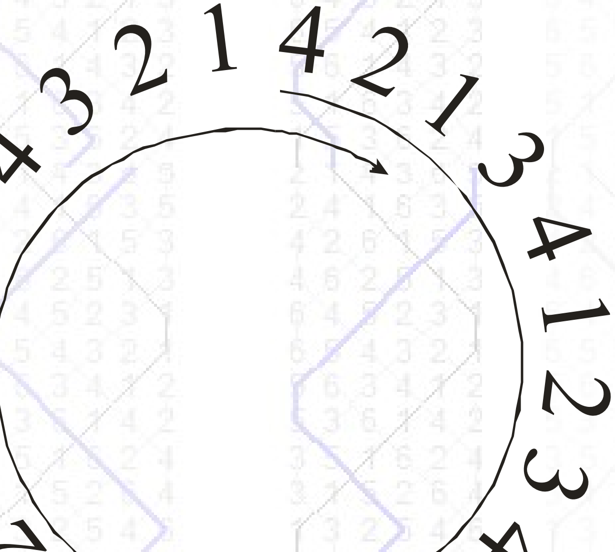
Theory and Applications

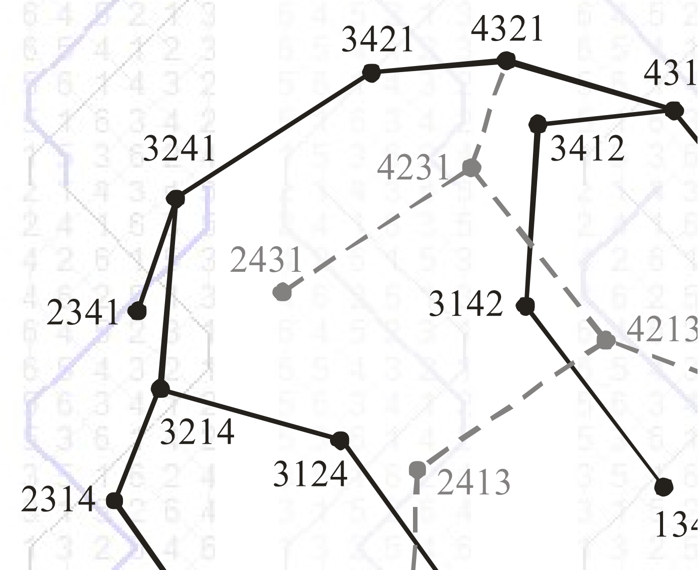
Alexander Holroyd
Microsoft Research

Frank Ruskey
University of Victoria

Brett Stevens
Carleton University

Aaron Williams
Carleton University

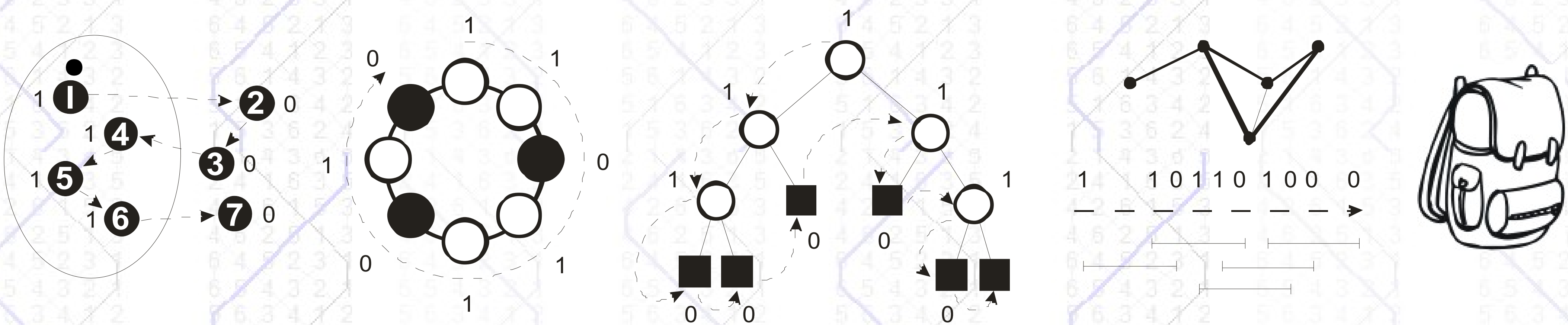




Combinatorial Generation

Orders of combinatorial objects and efficient algorithms that create these orders

- Objects often represented as strings
- Concentrate on objects of a fixed size and composition
- Connections to graph theory
- Efficient algorithms often reveal structure



Combinatorial objects represented by binary strings

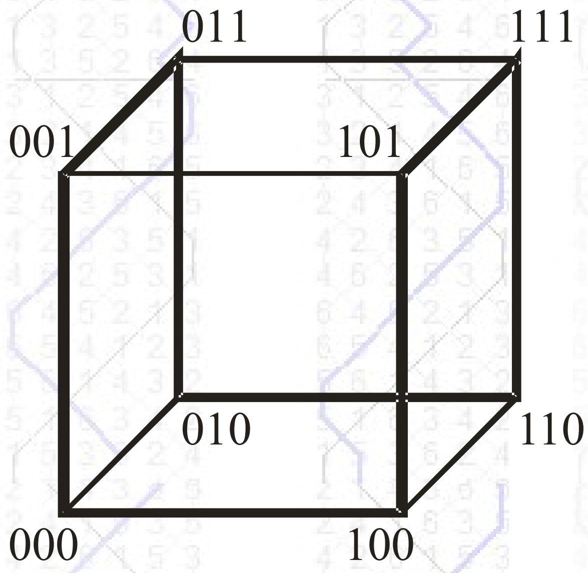
Combinatorial Generation

Historic Examples

- Binary reflected Gray code Patent 2,632,058 (1947)

000
001
011
010
110
111
101
100

BRGC for n=3

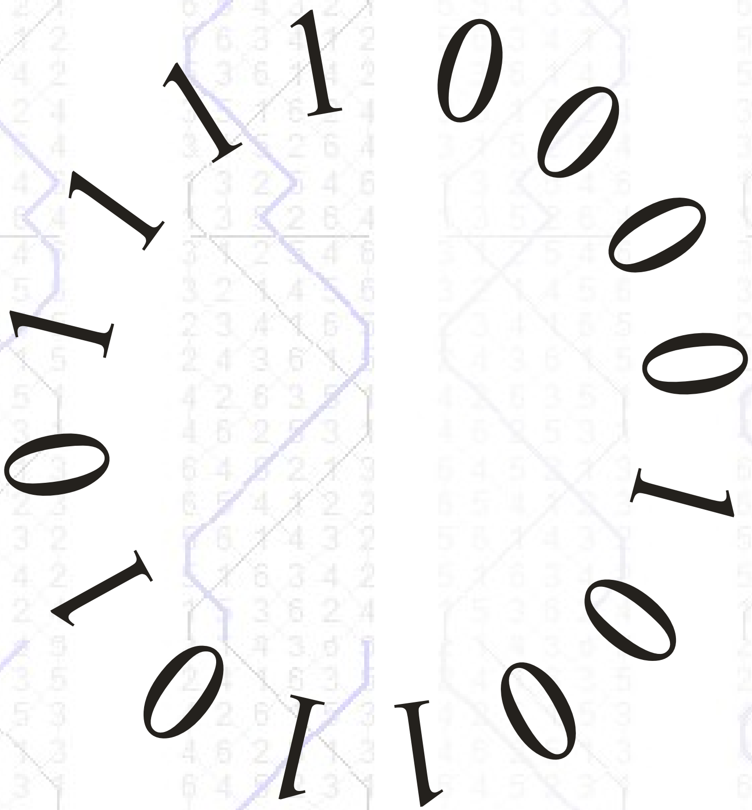


Hamilton cycle in the cube

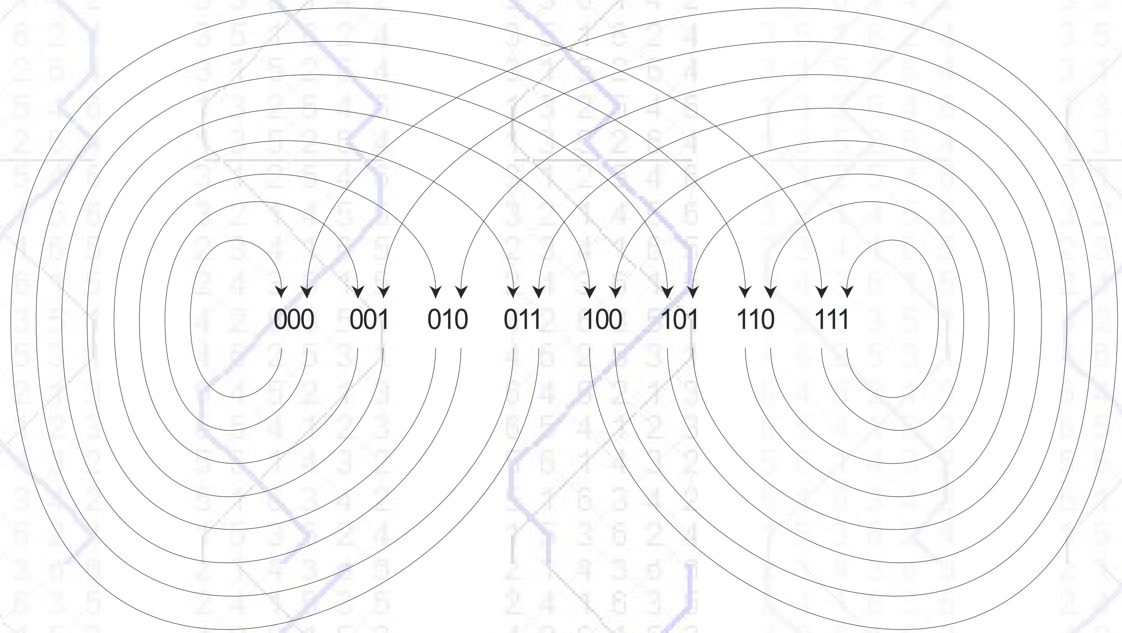
Combinatorial Generation

Historic Examples

- de Bruijn cycle (1946)



de Bruijn cycle for n=4



Eulerian cycle in the de Bruijn graph

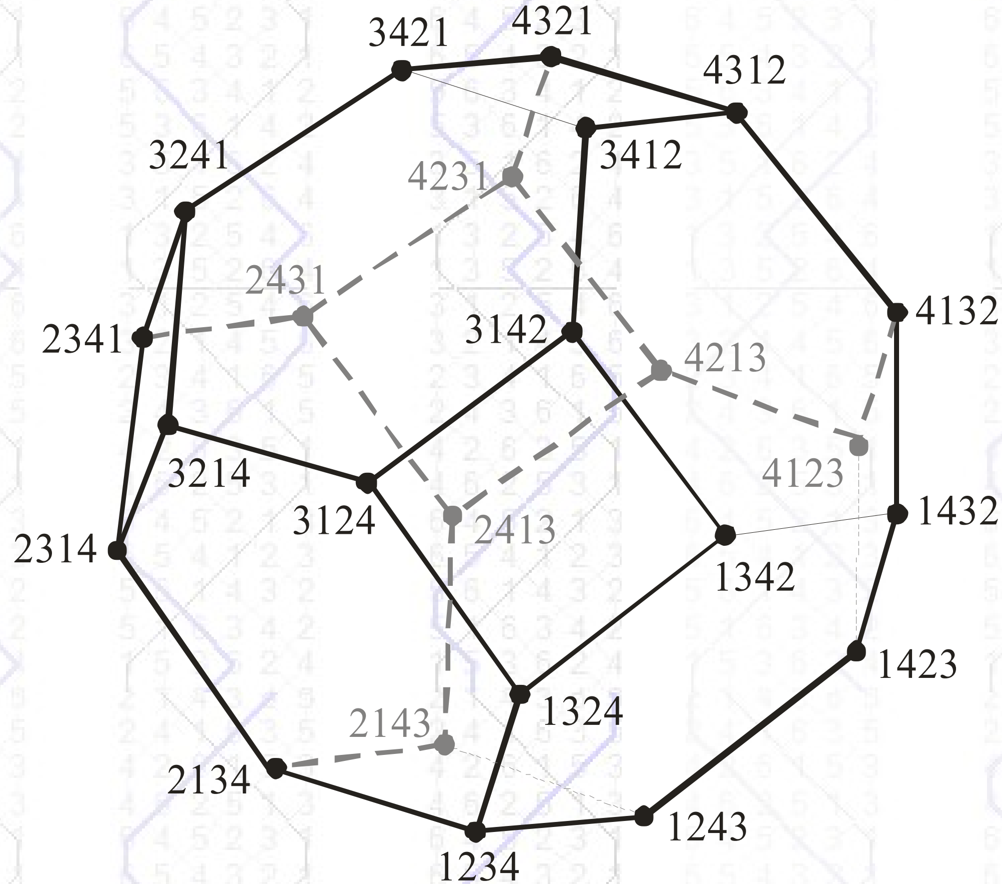
Combinatorial Generation

Historic Examples

- Johnson-Trotter-Steinhaus (1960s)

1234	**3124**	**2314**
1243	**3142**	**2341**
1423	**3412**	**2431**
4123	**4312**	**4231**
4132	**4321**	**4213**
1432	**3421**	**2413**
1342	**3241**	**2143**
1324	**3214**	**2134**

JTS for n=4

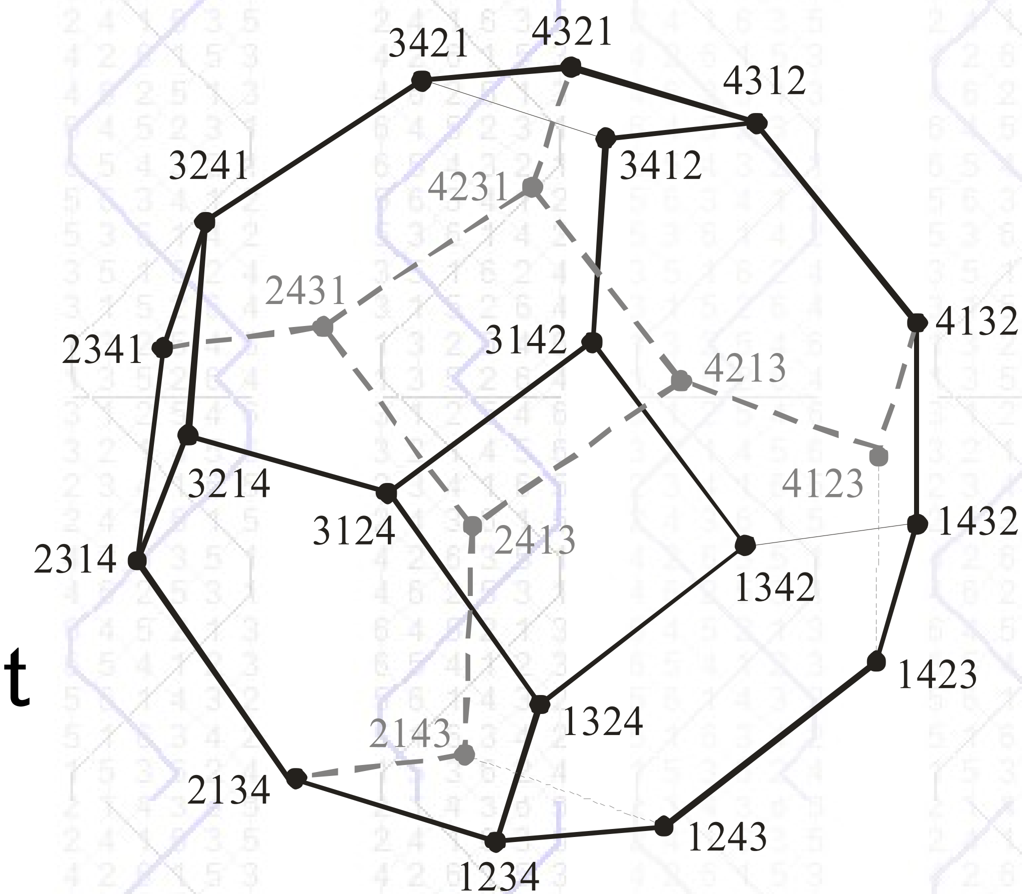


Hamilton cycle in the permutohedron

The Permutohedron

Definition

- Let $[n] = \{1,2,\ldots,n\}$
- Let τ_k be the adjacent-transposition $(k\ k+1)$
- Nodes are labeled by the permutations of $[n]$
- Edges between nodes that differ by τ_k for $k \in [n-1]$
- For example, 1234 is adjacent to 2134, 1324, and 1243

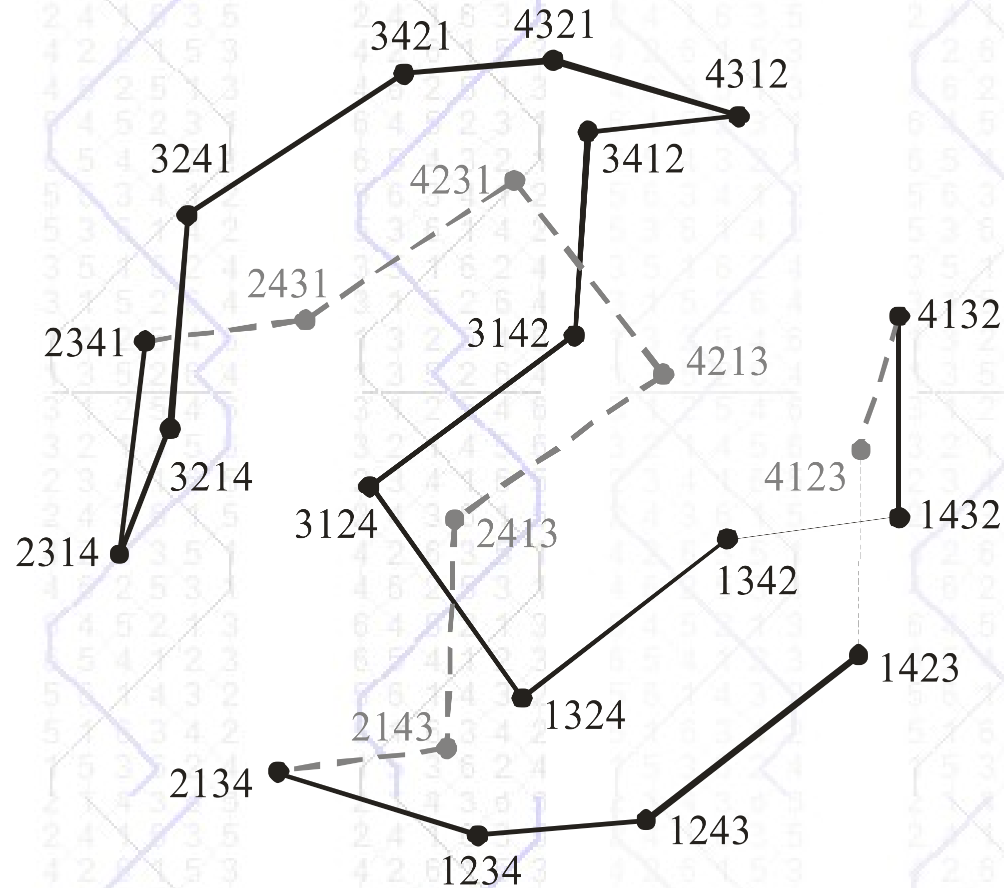


The Permutohedron for n=4

The Permutohedron

Hamilton Cycles

- Hamilton cycles are (cyclic) adj-transposition Gray codes
- Hamilton cycle uses a τ_k edge for each k ϵ [n-1]
- JTS generated by a loopless algorithm but successor is not trivial. For example, what follows 84253167?

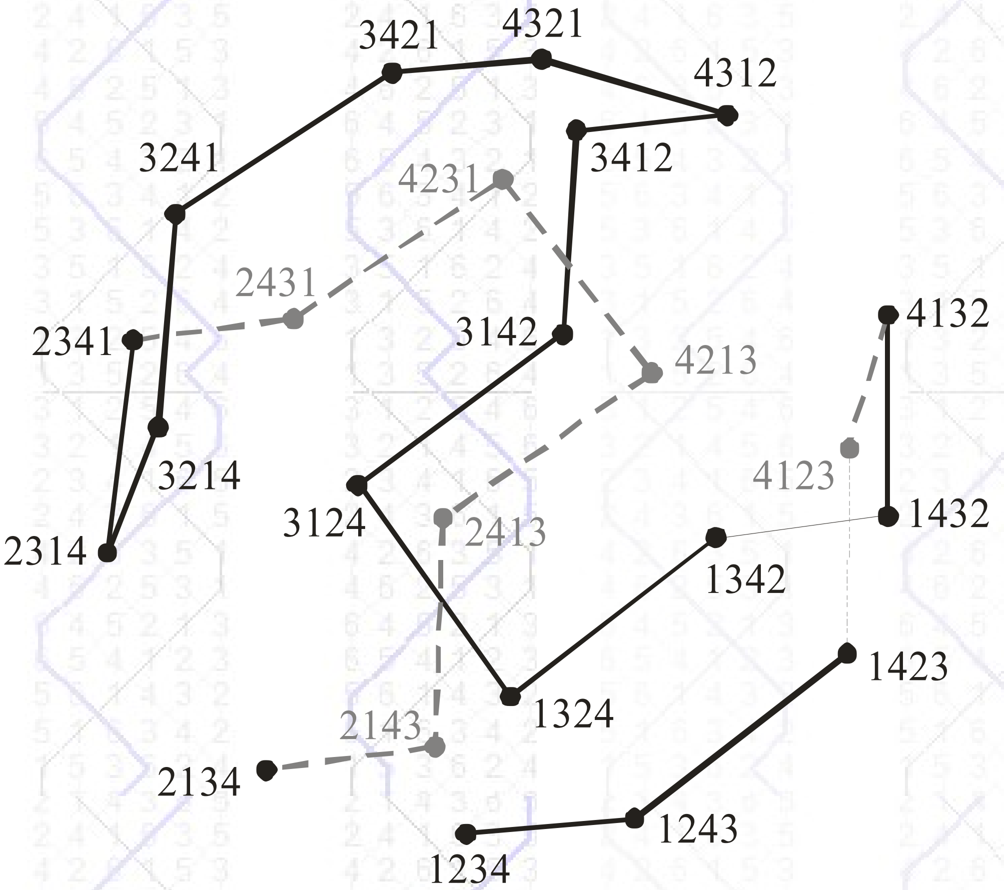


Johnson-Trotter-Steinhaus for n=4

The Permutohedron

Spanning Trees

- We will see that the spanning trees of the permutohedron are a special type of Gray code
- We define two spanning trees in which it is easy to determine each vertex's parent and children

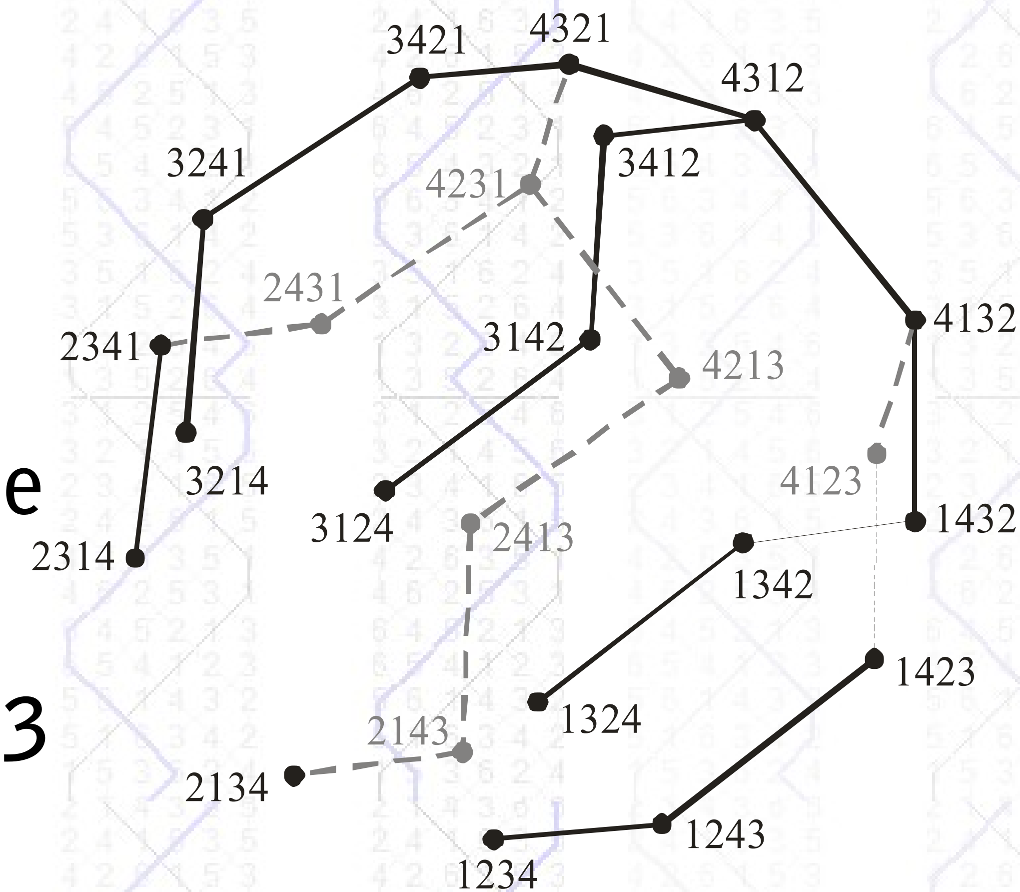


JTS Spanning Tree

The Permutohedron

Spanning Trees

- The *declining prefix* of a permutation of [n] is the longest prefix of the form n n-1 n-2 … j
- The *inclining symbol* is the symbol is j-1
- For example, in 87624153 declining prefix: 876 inclining symbol: 5

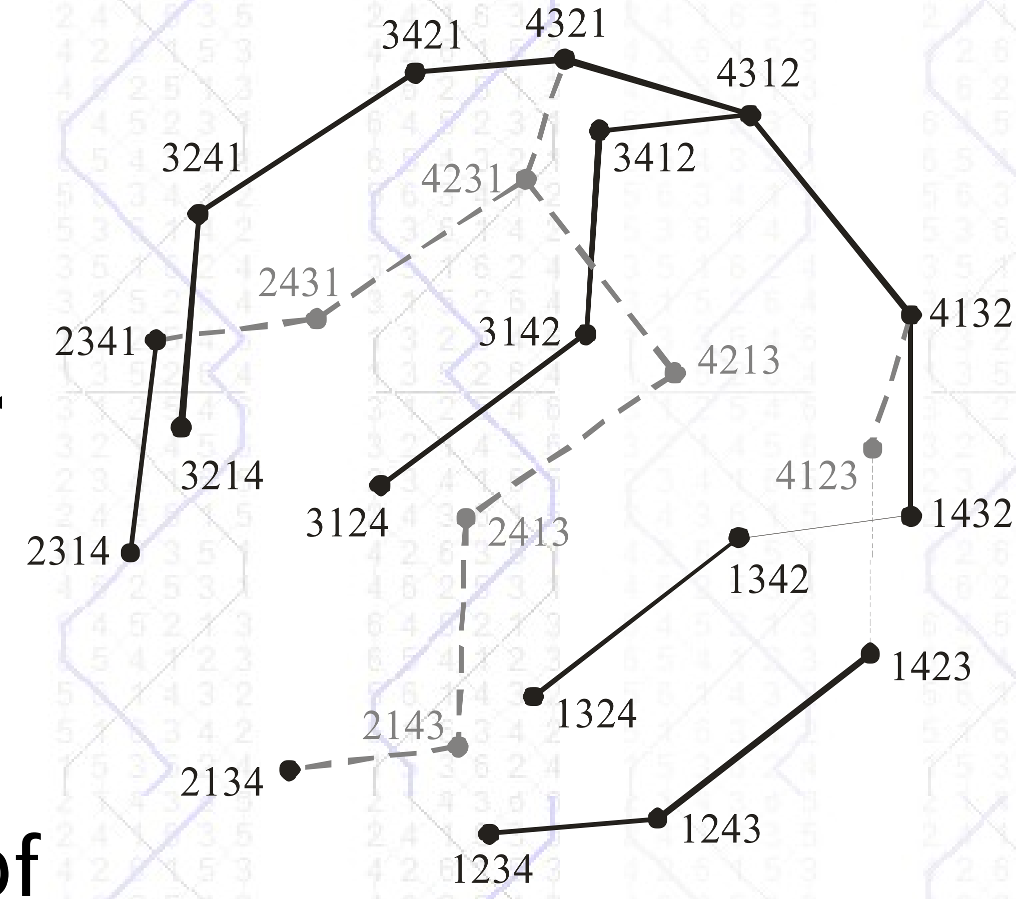


The Declining Spanning Tree

The Permutohedron

Spanning Trees

- The *declining spanning tree* is defined as follows:
- The root is n n-1 n-2 … 1
- The parent of every other vertex is obtained by swapping the inclining symbol to the left.
- For example, the parent of 87624153 is 87624513

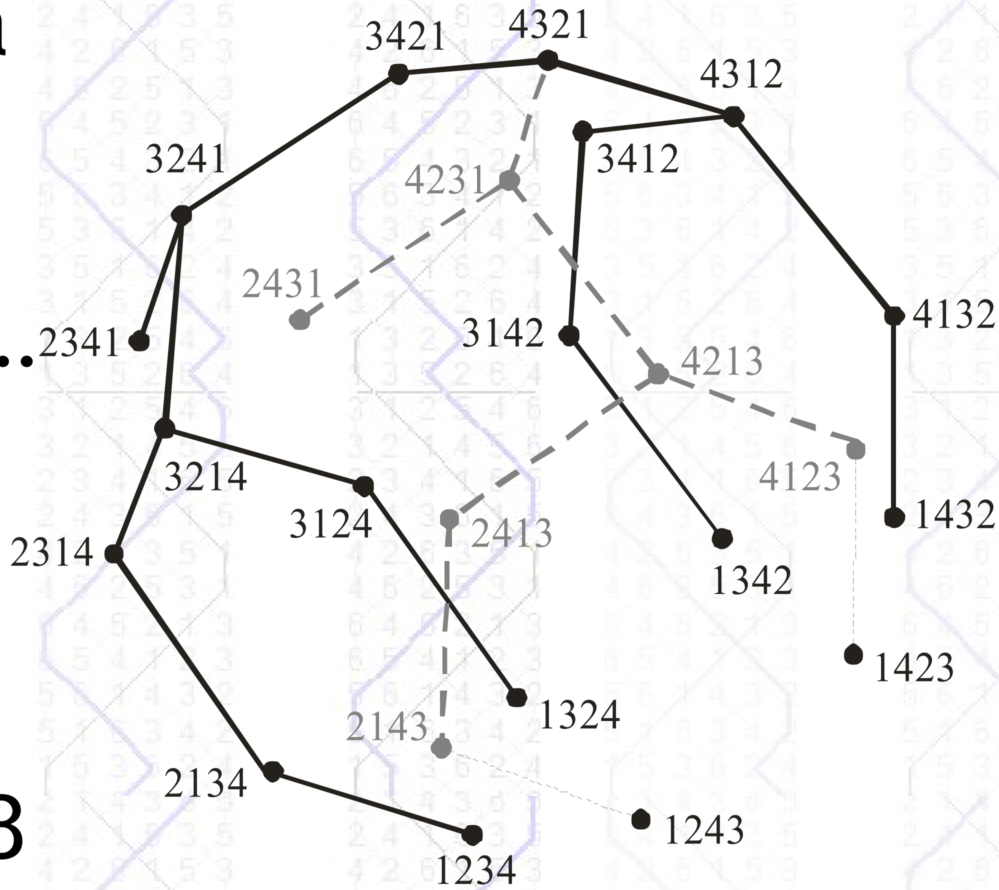


The Declining Spanning Tree

The Permutohedron

Spanning Trees

- The *decreasing prefix* of a permutation of [n] is the longest prefix of the form a b c … with a > b > c > …
- The *increasing symbol* is the symbol following the decreasing prefix
- For example, in 75326148 decreasing prefix: 7532 increasing symbol: 6

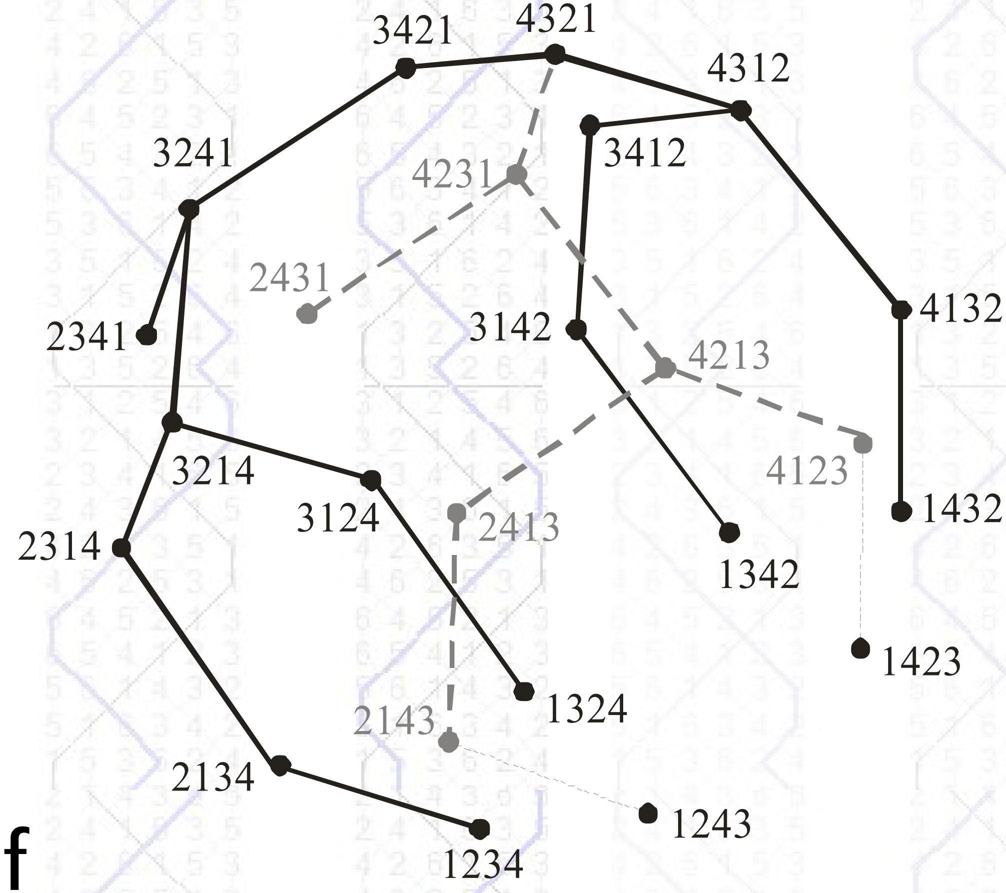


The Decreasing Spanning Tree

The Permutohedron

Spanning Trees

- The *decreasing spanning* tree is defined as follows:
- The root is n n-1 n-2 … 1
- The parent of every other vertex is obtained by swapping the increasing symbol to the left.
- For example, the parent of 75326148 is 75362148

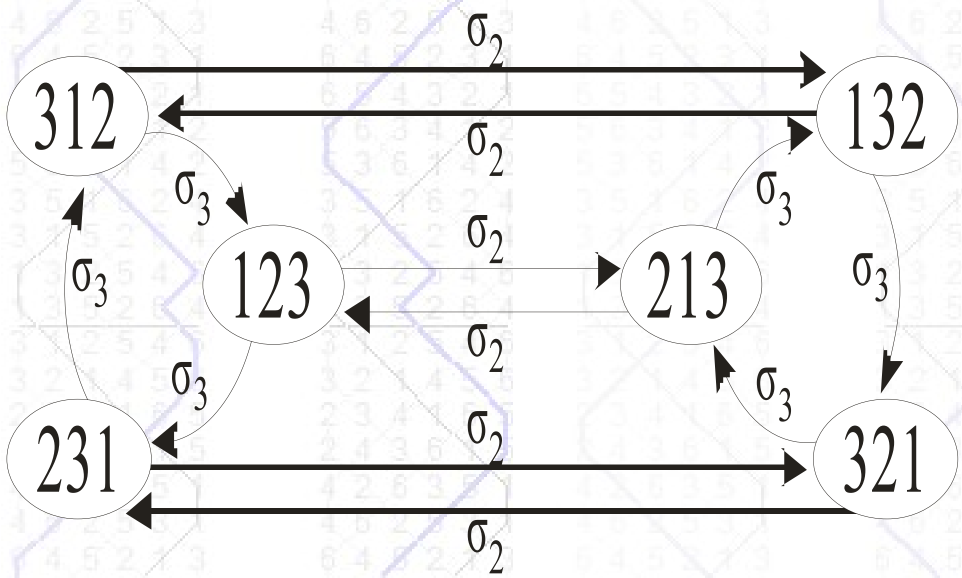


The Decreasing Spanning Tree

Rotator Graph

Definition

- Let σ_k be the prefix-shift (1 2 ... k)
- Nodes are labeled by the permutations of [n]
- Arcs directed from nodes to nodes that differ by σ_k for $k \in [n]$ (except for k=1)
- For example, 1234 has arcs directed to 2134, 2314, and 2341

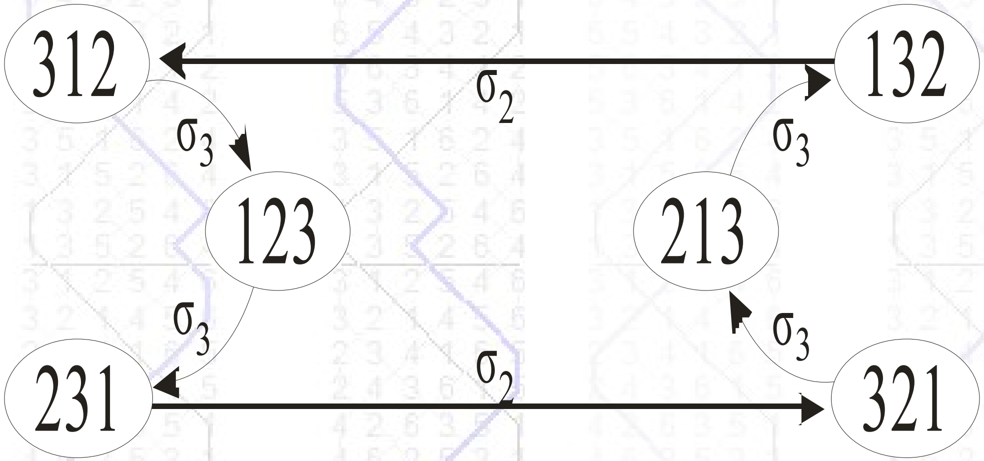


The rotator graph for n=3 is a particular Cayley graph

Rotator Graph

Hamilton Cycles

- Hamilton cycles are (cyclic) prefix-shift Gray codes.
- Hamilton cycles do not necessarily use a σ_k arc for each $k \in [n]$
- Restricted rotator graphs use a subset of $\sigma_2 \sigma_3 \ldots \sigma_n$
- Do they exist?



Hamilton cycle

Combinatorial Generation

Prefix-Shift Gray Codes

- Corbett (1992) used the rotator graph for point-to-point multiprocessor networks

321	3
213	3
132	2
312	3
123	3
231	2

Corbett for n=3

4321	4	**4213**	4	**4123**	4
3214	4	**2134**	4	**1234**	4
2143	4	**1342**	4	**2341**	4
1432	2	**3421**	3	**3412**	2
4132	4	**4231**	4	**4312**	4
1324	4	**2314**	4	**3124**	4
3241	4	**3142**	4	**1243**	4
2413	2	**1423**	2	**2431**	3

Corbett for n=4

Combinatorial Generation

Prefix-Shift Gray Codes

- W (2009) cool-lex order. First symbol a is shifted past or between the first bc with b<c. First case if a>b and second case if a>b. If no such bc then past last symbol.

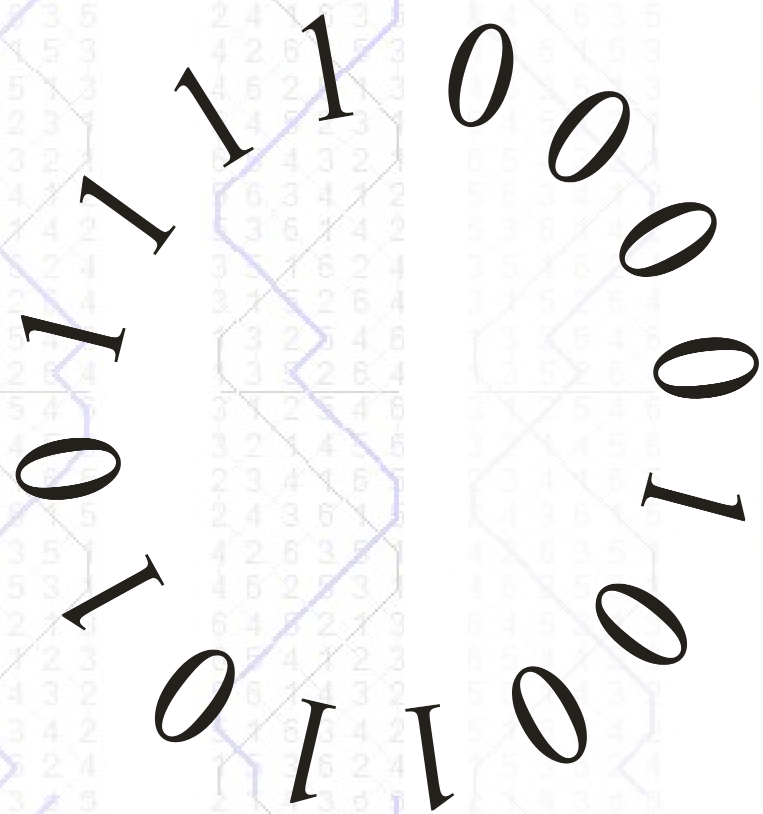
4321	**2341**	**4231**
3214	**3421**	**2431**
2134	**4213**	**4312**
1234	**2143**	**3142**
2314	**1243**	**1342**
3124	**2413**	**3412**
1324	**4123**	**4132**
3241	**1423**	**1432**

cool-lex for n=4

Universal Cycles

Definition

- A *universal cycle* is a circular string containing each string in a set L exactly once as a substring
- If there are no universal cycles for L then a simple *encoding* of each string in L can be considered
- *Decoding* a universal cycle gives its "Gray code" of L

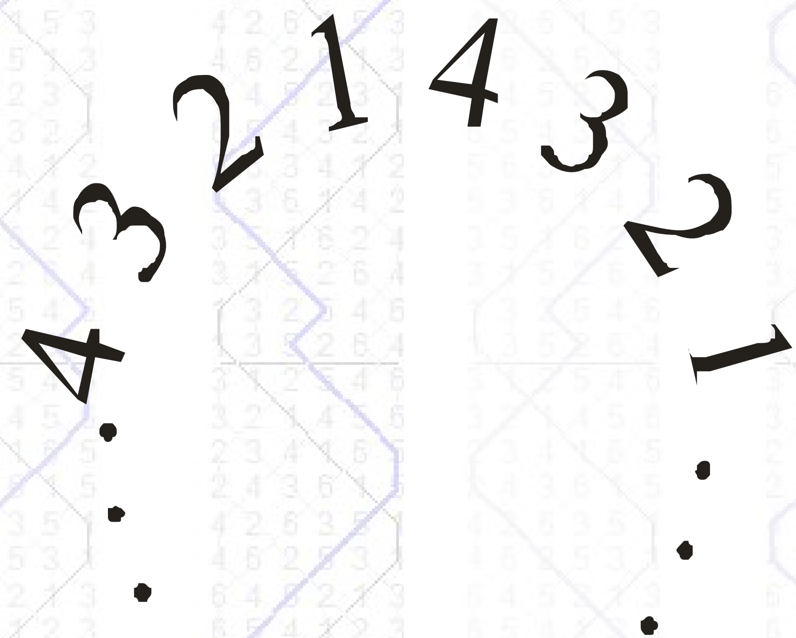


A de Bruijn cycle for the binary strings of length 4

Universal Cycles

Permutations

- Universal cycles for the permutations of [n] do not exist when n > 2

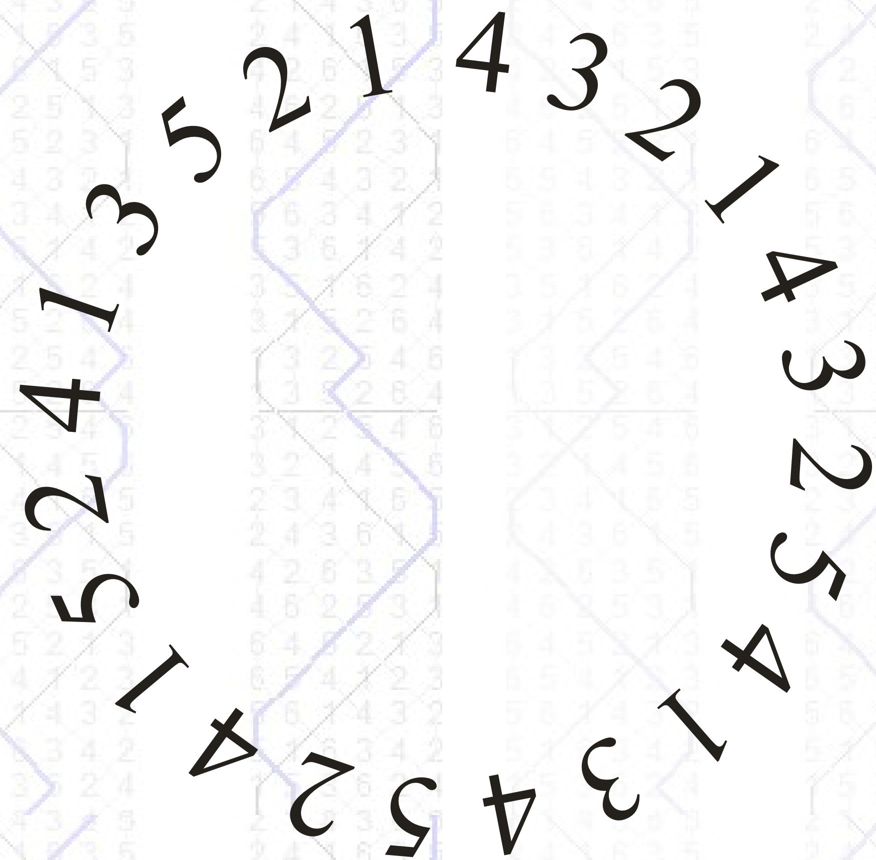


A single permutation is forced to repeat

Universal Cycles

Permutations using Relative Order

- Permutations can be encoded by *relative order*
- For example, 5143 has the relative order of 4132
- Johnson proved n+1 symbols are sufficient for the permutations of [n]
- However, the "Gray code" is not a Gray code

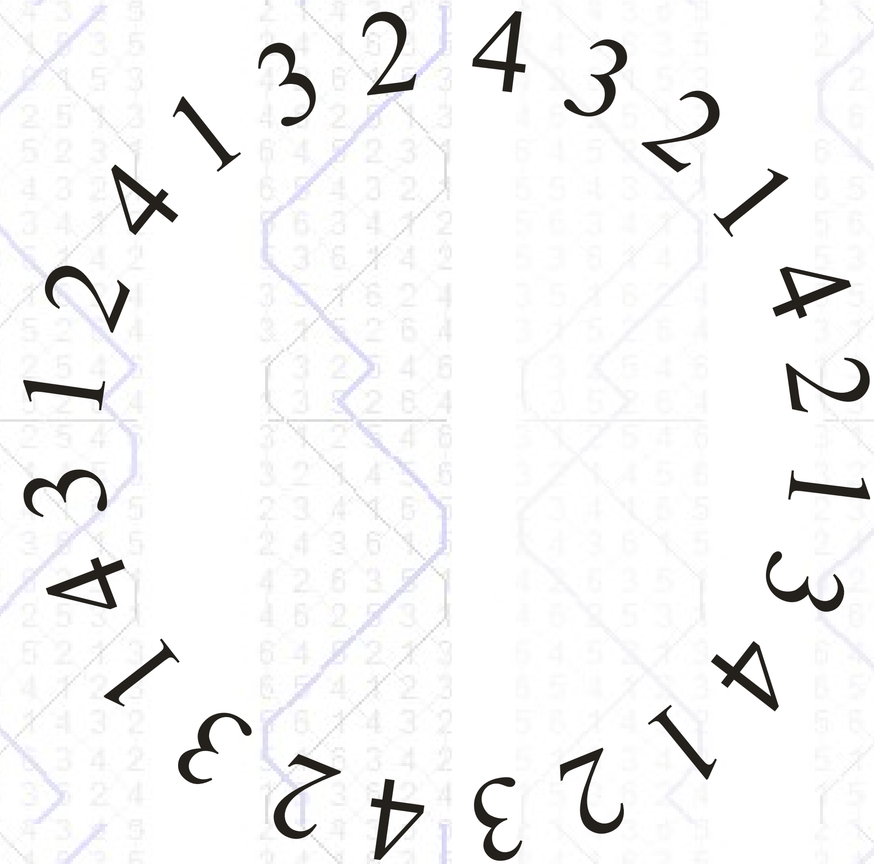


Each permutation of [4] is encoded by relative order

Universal Cycles

Permutations using Shorthand

- Permutations can be encoded by *shorthand*
- For example, 413 is shorthand for 4132
- Shorthand encodings are the (n-1)-perms of [n]
- Jackson proved universal cycles exist for the k-perms of [n] when k < n
- Efficient Construction? -Knuth

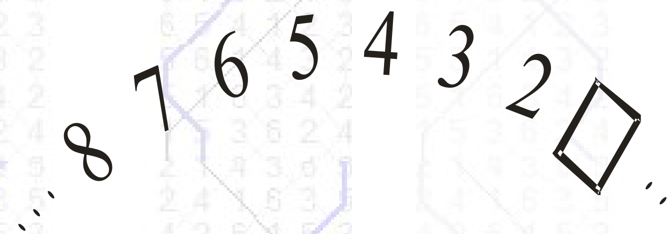


Each permutation of [4] is encoded by shorthand

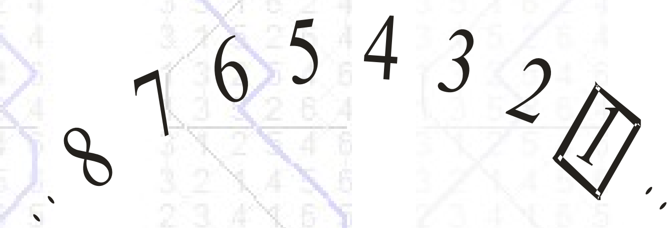
Shorthand Universal Cycles for Permutations

Gray Code using σ_n / σ_{n-1}

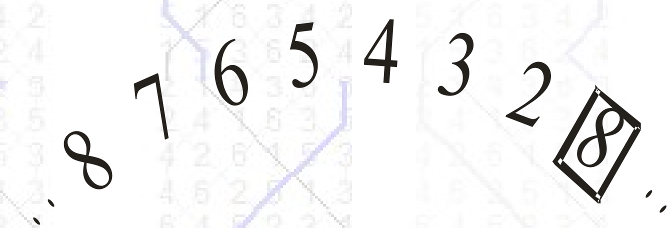
- Each shorthand substring is followed by its missing symbol or its first symbol
- Permutations differ by a prefix shift σ_n / σ_{n-1}
- Gray codes using σ_n / σ_{n-1} correspond to shorthand universal cycles for permutations



The next symbol is 1 or 8



87654321 followed by 76543218
prefix shift of length 8

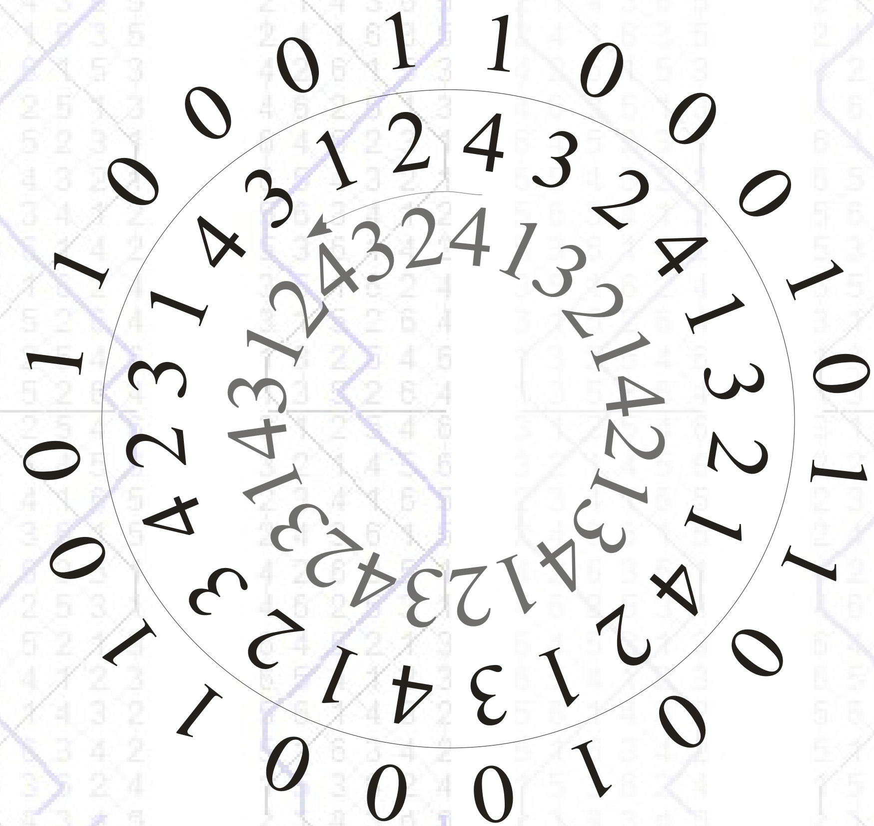


87654321 followed by 76543281
prefix shift of length 7

Shorthand Universal Cycles for Permutations

Binary Representation

- Due to the σ_n / σ_{n-1} Gray code the cycle can be represented by n! bits 0 / 1
- A cycle is *max-weight* or *min-weight* if the sum of its binary representation is the maximum or minimum among all such cycles

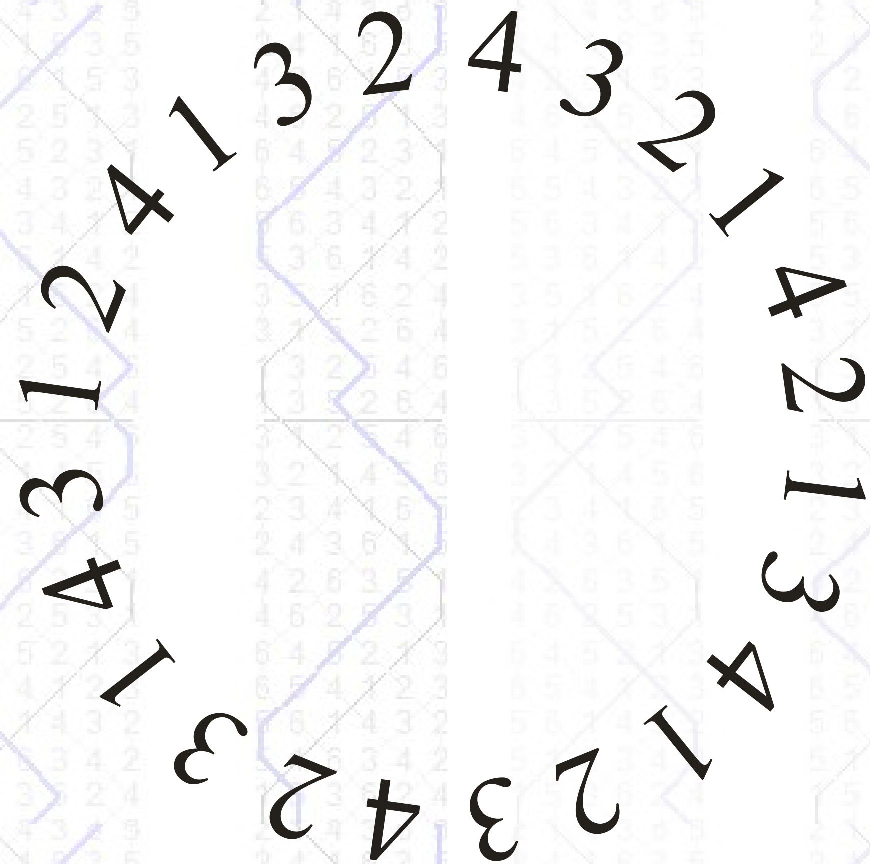


Binary representation (outer) for the universal cycle (inner)

Shorthand Universal Cycles for Permutations

Periodic

- If n appears as every nth symbol then the cycle is *periodic*
- The *sub-permutations* appear between each copy of n and are the permutations of [n-1]

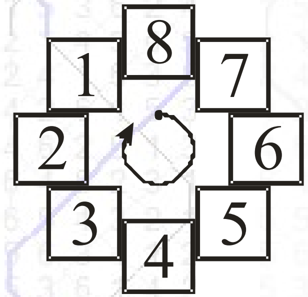


A periodic shorthand universal cycle of the form 4…4…4…4…4…4…

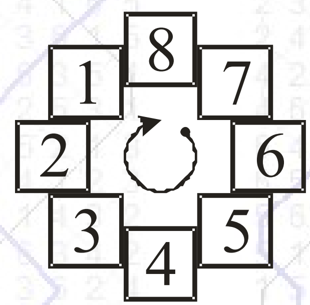
Shorthand Universal Cycles for Permutations

Applications

- The σ_n / σ_{n-1} operations are efficient in linked lists and circular arrays
- The σ_n / σ_{n-1} Gray code allows faster exhaustive solutions to Traveling Salesman problems like Shortest Hamilton Path
- Min-weight is desirable

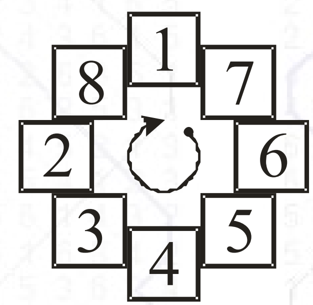


87654321
is followed by



76543218
increment start

or

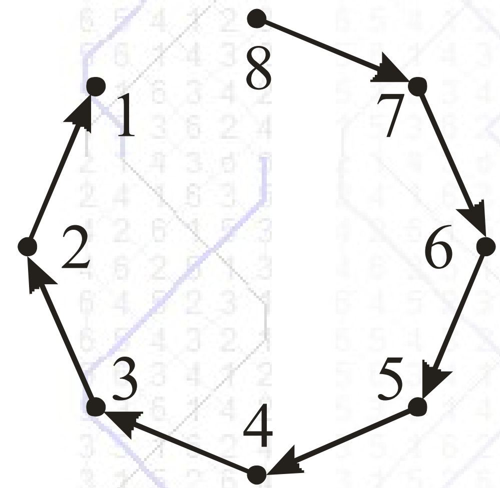


76543281
increment start
adj-transposition

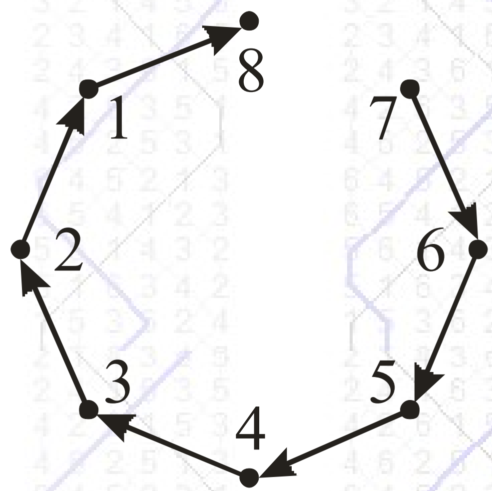
Shorthand Universal Cycles for Permutations

Applications

- The σ_n / σ_{n-1} operations are efficient in linked lists and circular arrays
- The σ_n / σ_{n-1} Gray code allows faster exhaustive solutions to Traveling Salesman problems like Shortest Hamilton Path
- Min-weight is desirable

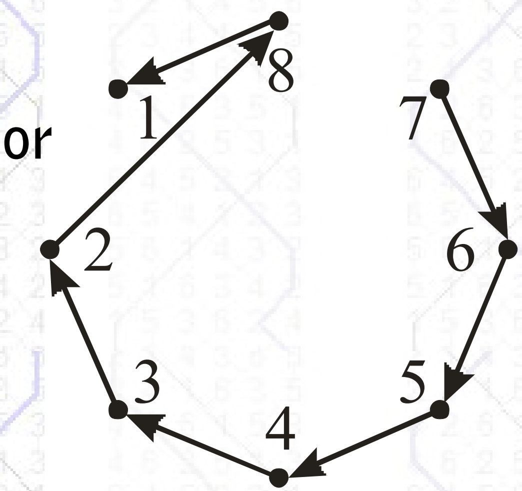


Path 87654321 followed by ...



Path 76543218
-87 +18

or

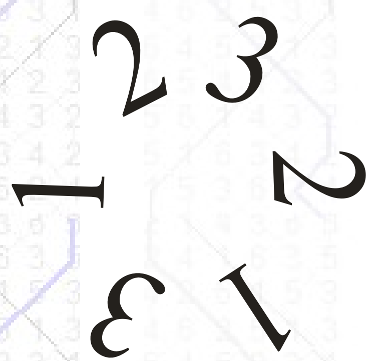


Path 76543281
-87 -21 +28 +81

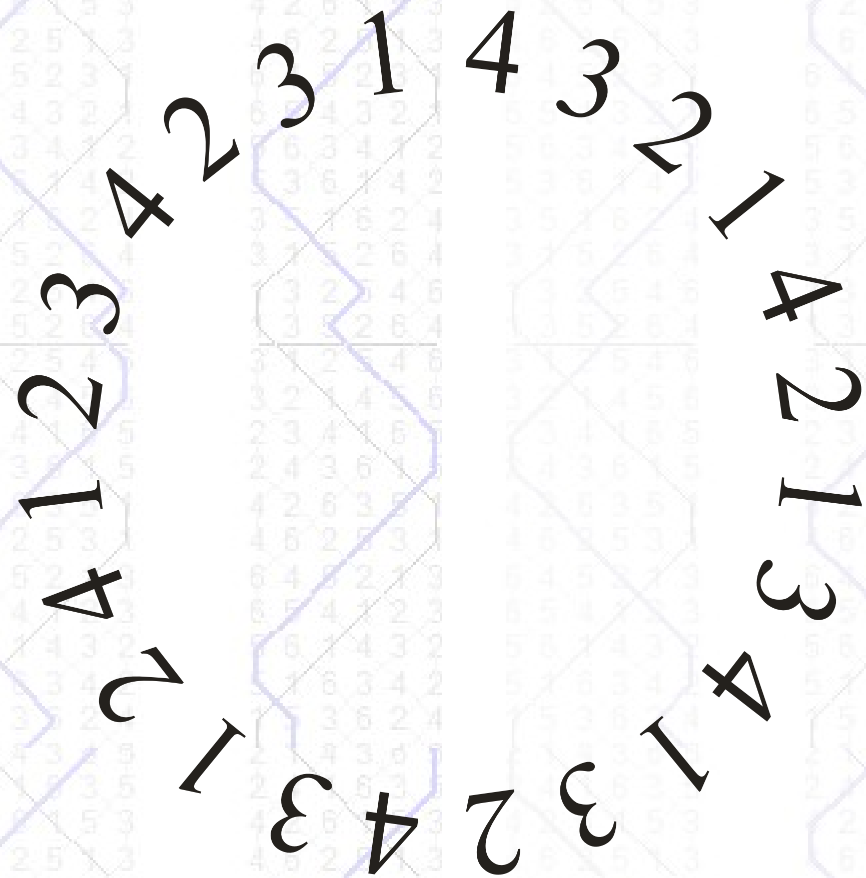
Constructions

Recycling

- The Gray code of a cycle for [n] are sub-permutations of a cycle for [n+1]



Gray code
321, 213, 132, 312, 123, 231

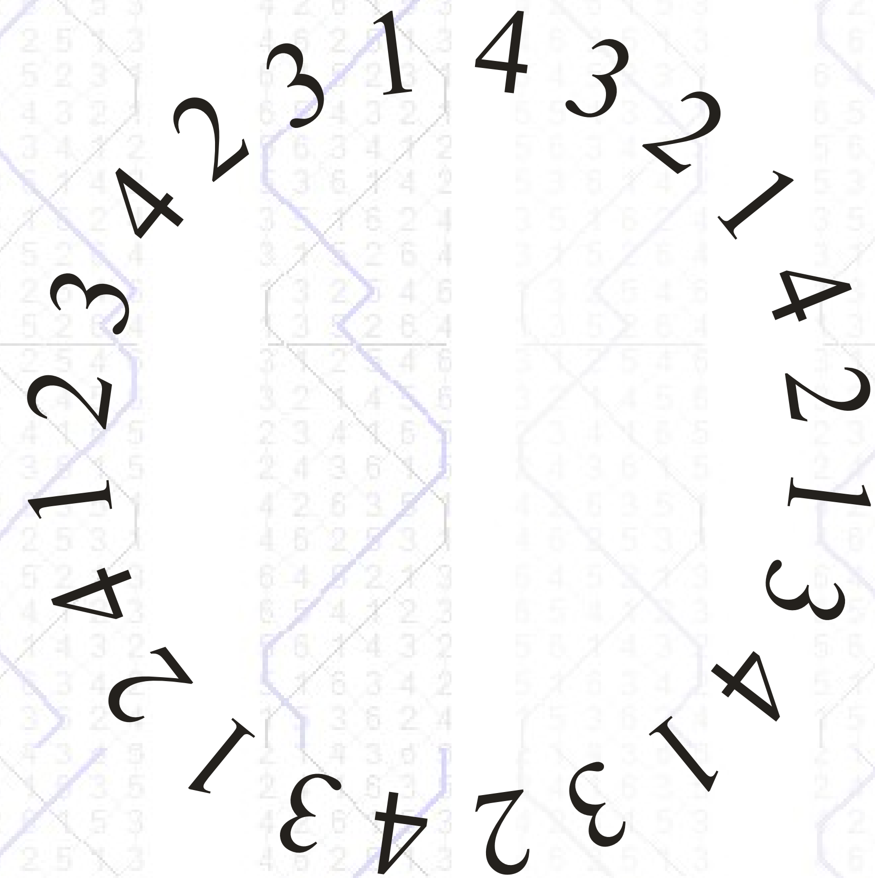


Sub-permutations
321, 213, 132, 312, 123, 231

Constructions

Recycling

- Any σ_n / σ_{n-1} Gray code for [n] are sub-permutations of a periodic cycle for [n+1]
- This helped answer the question of Knuth's for an efficient construction by Ruskey-Williams
- *Recycling* inserts the symbol n+1 between a given order of permutations of [n]



Sub-permutations
321, 213, 132, 312, 123, 231

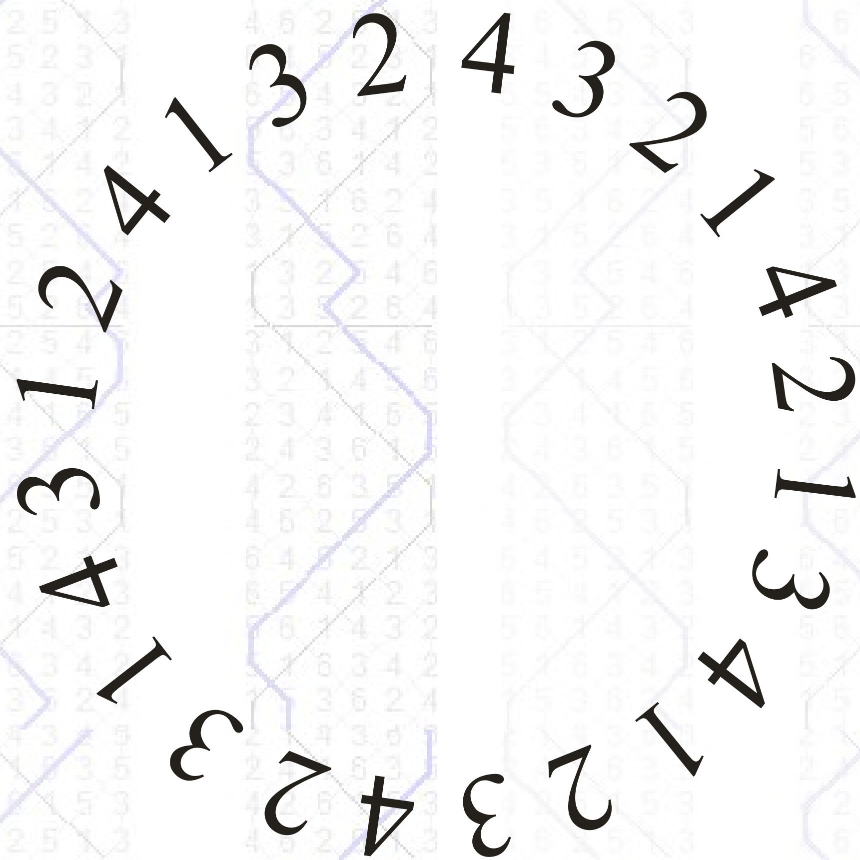
Constructions

Recycling

- Holroyd-Ruskey-W proved cool-lex / *7-order* recyclable
- Stevens-W proved Corbett's order is recyclable
- Most orders not recyclable; adj-transposition Gray codes
- Cool-lex and 7-order give min-weight periodic cycles

321, 213, 123, 231, 312, 132

cool-lex order

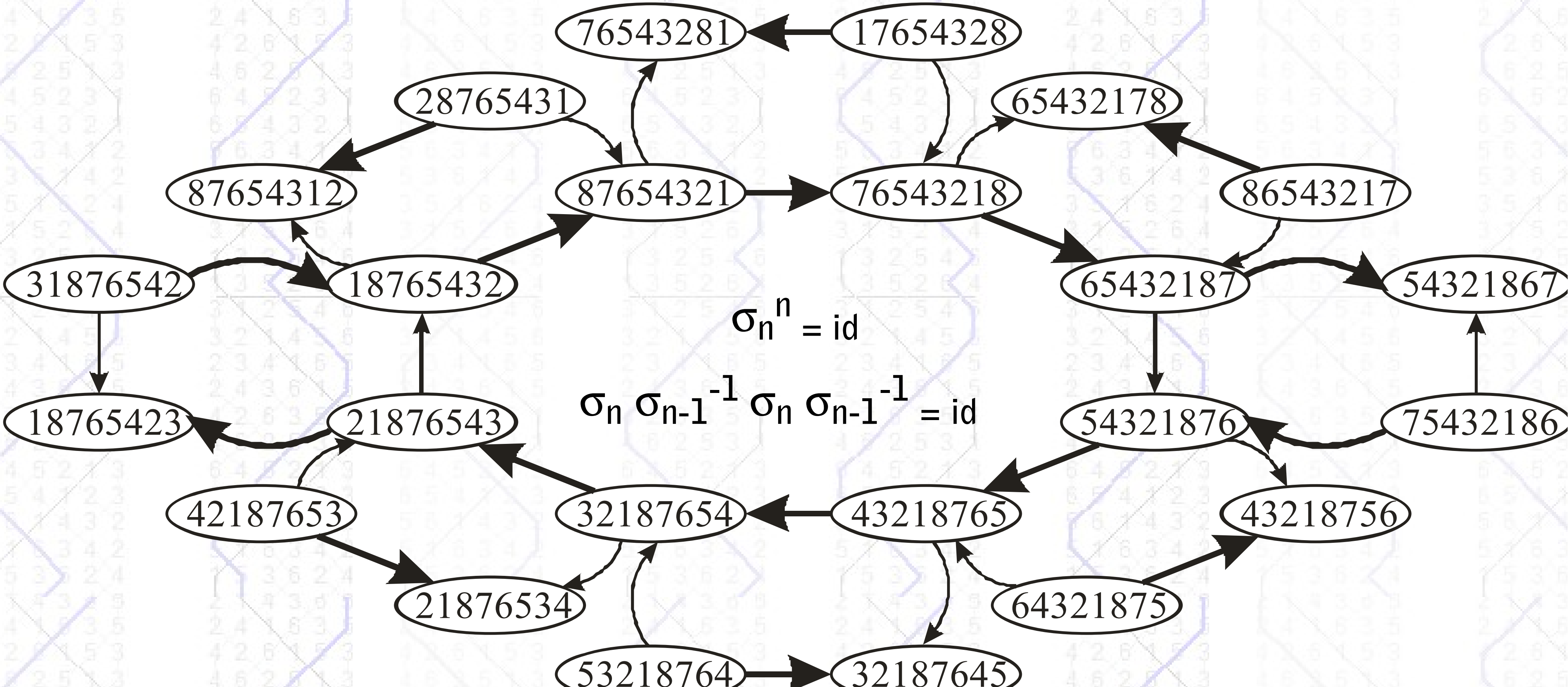


Sub-permutations

321, 213, 123, 231, 312, 132

Min-Weight Periodic Cycles

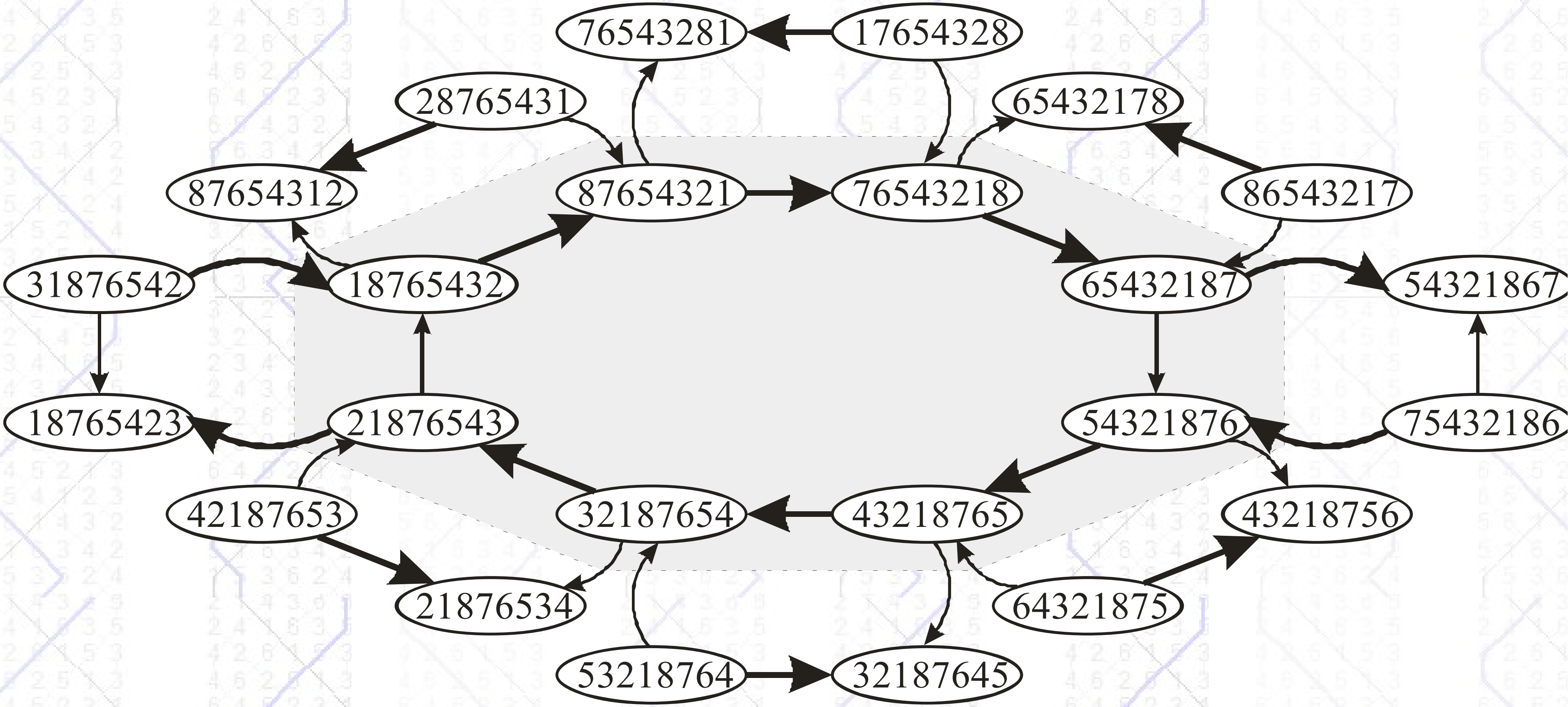
A Local View...



The Cayley graph with generators σ_n / σ_{n-1} has alternating cycles of length four. If C is a Hamilton cycle, then the arcs on each 4-cycle satisfy: (i) both σ_n arcs in C and both σ_{n-1} arcs not in C, or (ii) both σ_n arcs not in C and both σ_{n-1} arcs in C.

Min-Weight Periodic Cycles

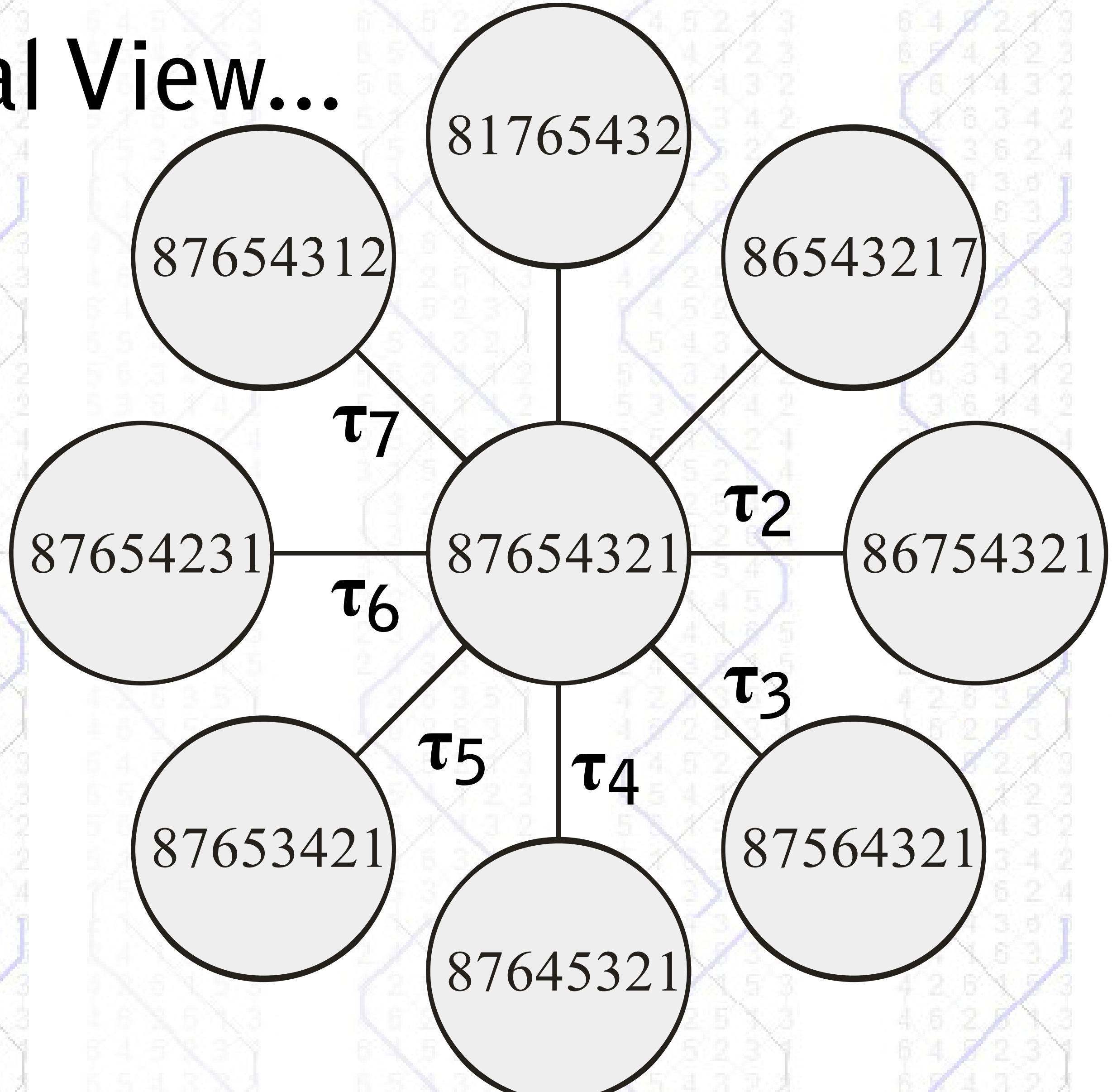
A Local View...



Consider the contraction of each σ_n^n cycle.

Min-Weight Periodic Cycles

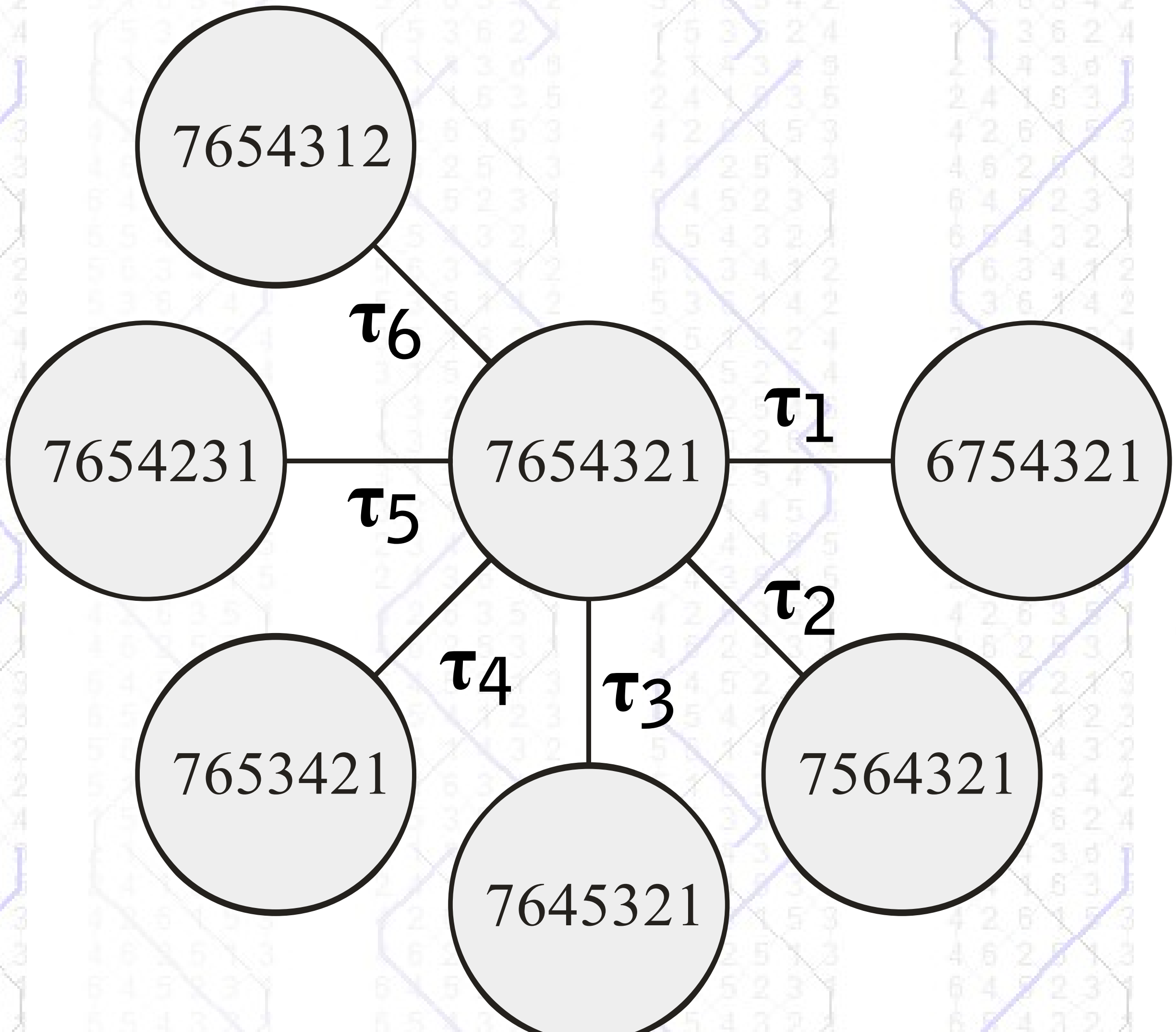
A Local View...



Label nodes with rotation starting with n.

Min-Weight Periodic Cycles

A Local View...



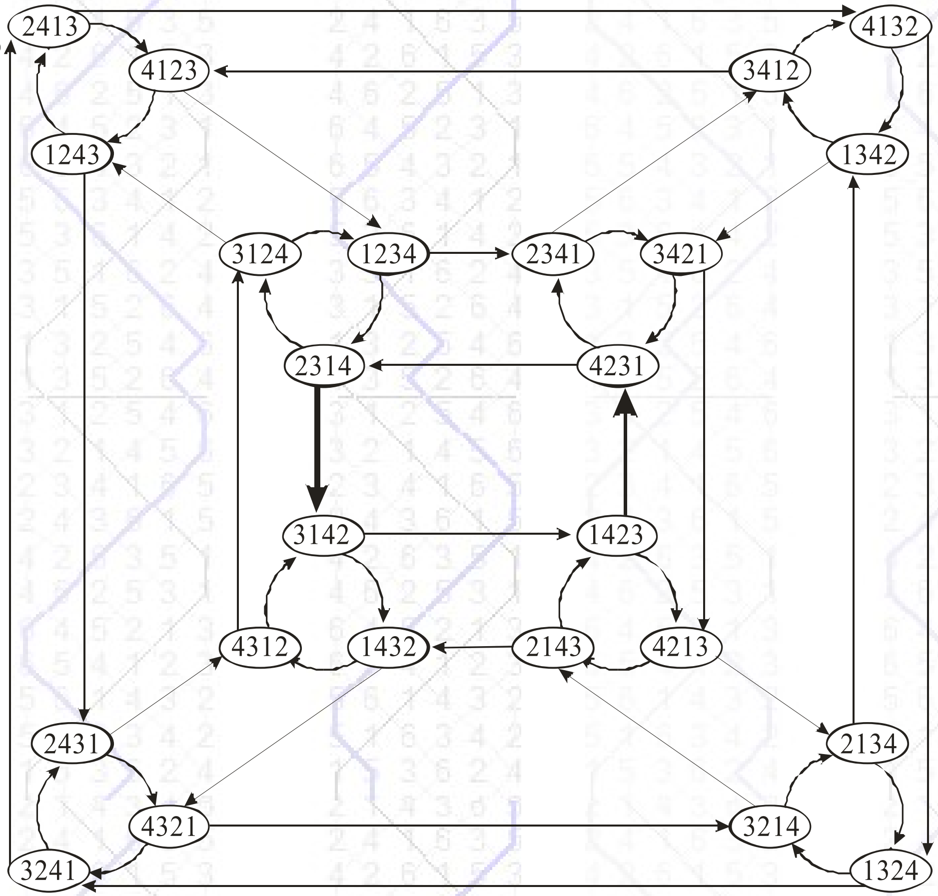
Remove two edges and relabel to get the permutohedron.

Note: The universal cycle is periodic iff it does not use these two types of edges.

Min-Weight Periodic Cycles

A Global View...

- Shorthand universal cycles are Hamilton cycles in the σ_n / σ_{n-1} Cayley graph

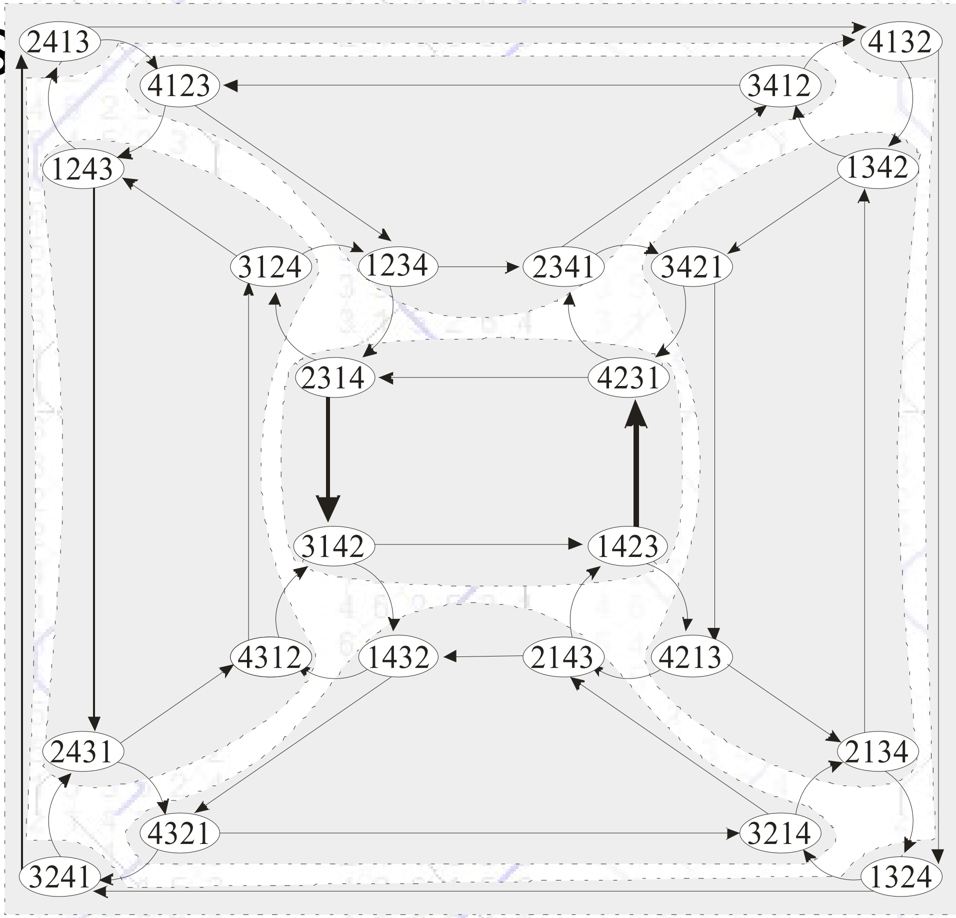


Cayley graph of the symmetric group for n=4 with generators σ_n / σ_{n-1}

Min-Weight Periodic Cycles

A Global View…

- Shorthand universal cycles are Hamilton cycles in the σ_n / σ_{n-1} Cayley graph
- Hamilton cycles enter/exit each σ_n^n cycle so after contraction min-weight Ham cycles=spanning tree

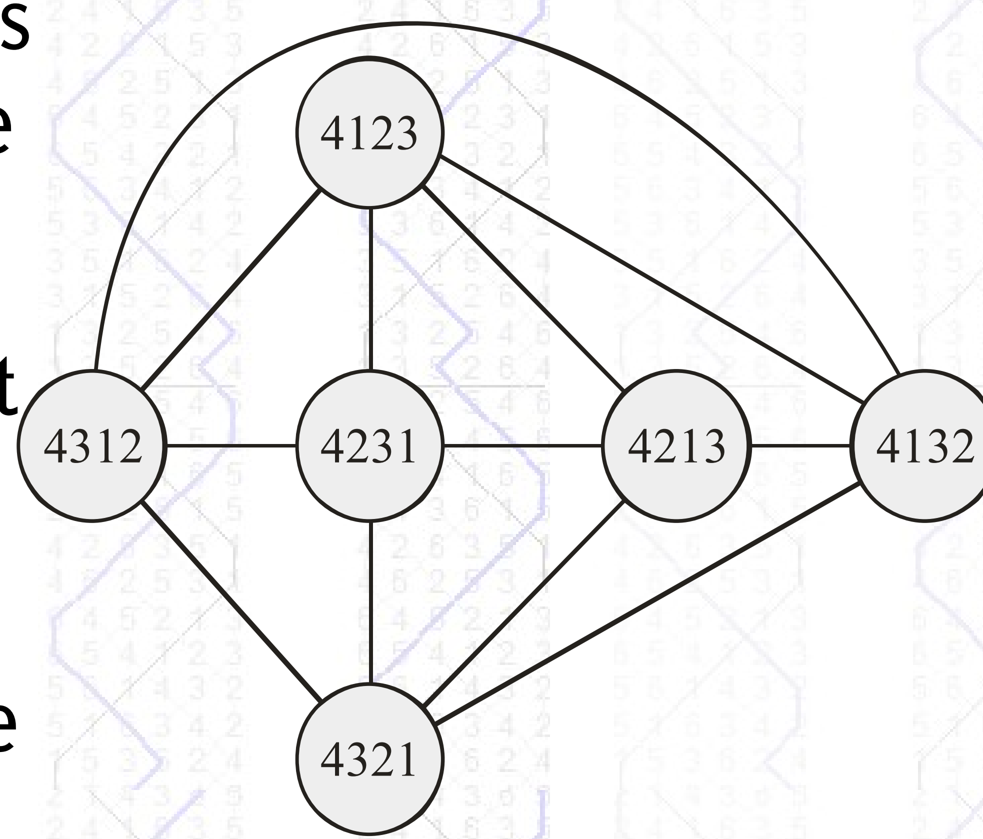


Directed cycles using σ_n are contracted

Min-Weight Periodic Cycles

A Global View...

- Shorthand universal cycles are Hamilton cycles in the σ_n / σ_{n-1} Cayley graph
- Hamilton cycles enter/exit each σ_n^n cycle so after contraction min-weight Ham cycles=spanning tree

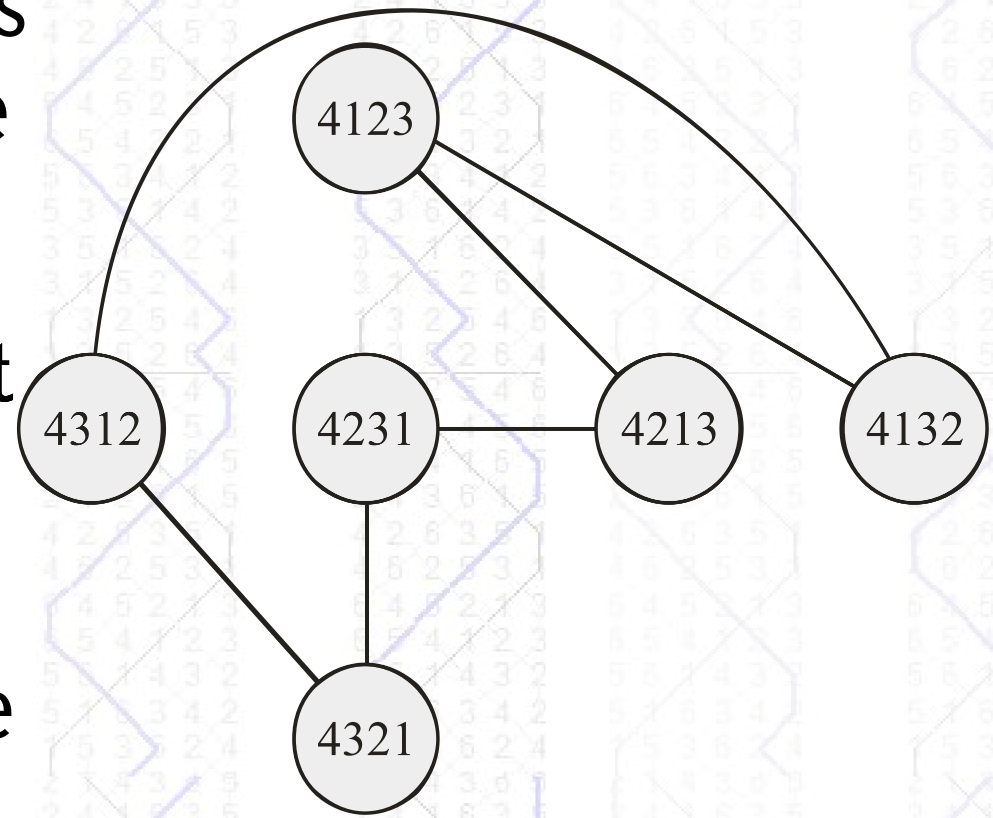


Directed cycles using σ_n are contracted

Min-Weight Periodic Cycles

A Global View…

- Shorthand universal cycles are Hamilton cycles in the σ_n / σ_{n-1} Cayley graph
- Hamilton cycles enter/exit each σ_n^n cycle so after contraction min-weight Ham cycles=spanning tree

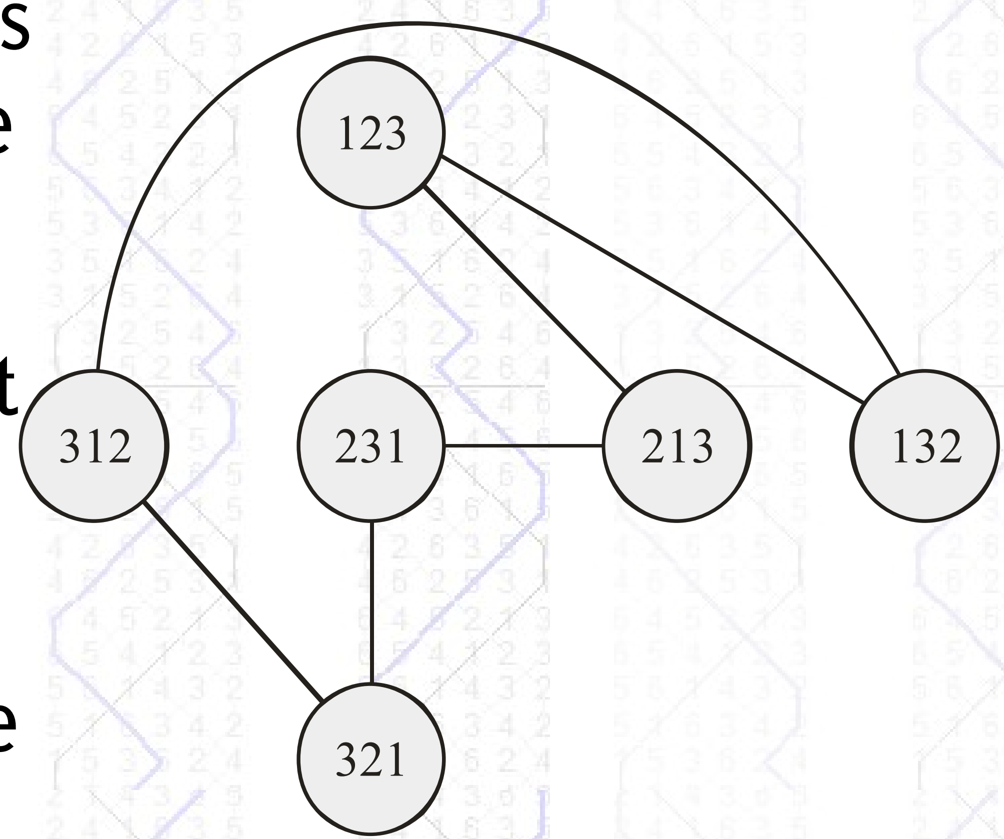


Edges are removed

Min-Weight Periodic Cycles

A Global View...

- Shorthand universal cycles are Hamilton cycles in the σ_n / σ_{n-1} Cayley graph
- Hamilton cycles enter/exit each σ_n^n cycle so after contraction min-weight Ham cycles=spanning tree
- Remove arcs to get the permutohedron and its spanning trees are periodic

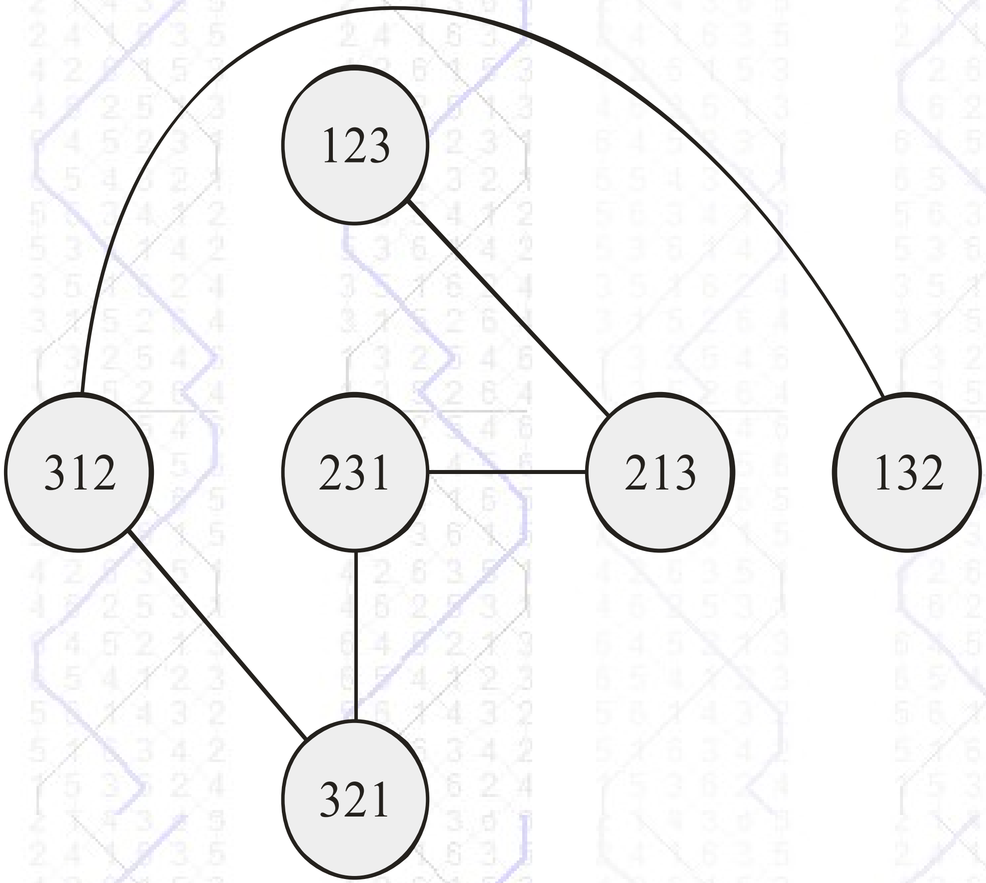


Vertices relabeled to obtain the permutohedron for n-1

Min-Weight Periodic Cycles

Characterization

- Theorem: There is a simple bijection between the min-weight periodic shorthand universal cycles for permutations of [n] and spanning trees of the permutohedron for [n-1]

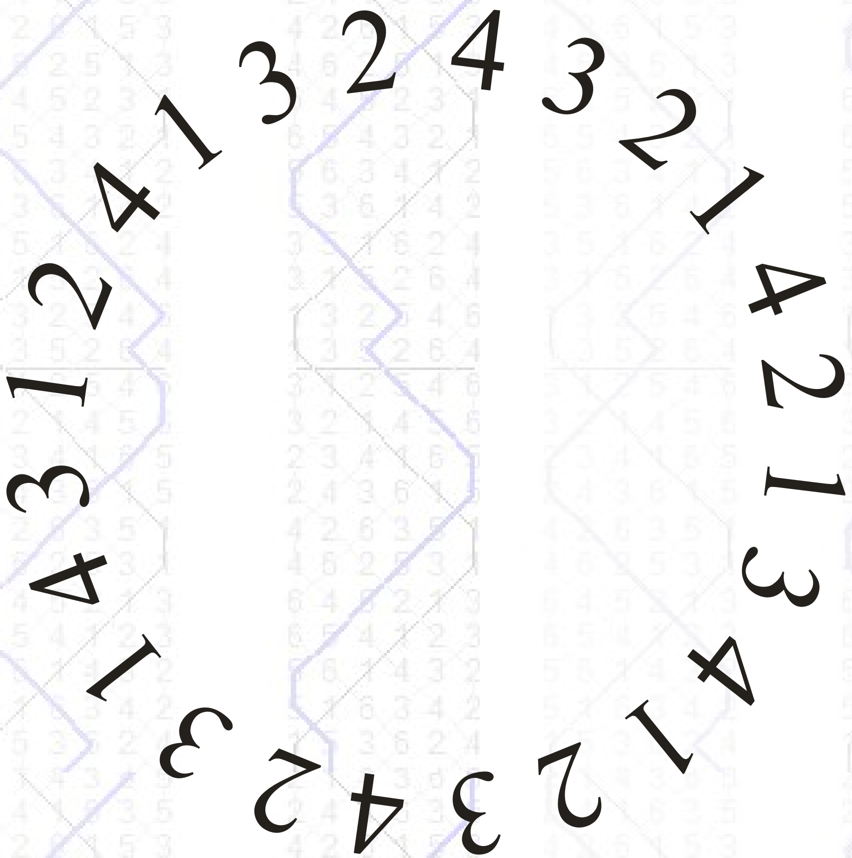


The decreasing spanning tree for the permutohedron for [3]

Min-Weight Periodic Cycles

Characterization

- Theorem: There is a simple bijection between the min-weight periodic shorthand universal cycles for permutations of [n] and spanning trees of the permutohedron for [n-1]



The shorthand universal cycle for the permutations of [4] from cool-lex order

Memoryless Gray Code Algorithm

The Gray codes for the cool-lex and 7-order shorthand universal cycles for permutations can be created by simple memoryless rules

Theorem 7 *Suppose* $\mathbf{a} \in \Pi(n)$ *where* $m = \max(a_1, a_n)$ *and* d *is the minimum value in its decrementing substring. The permutation of* $\mathsf{B}(n)$ *that follows* $\mathbf{a}$ *is*

- $\mathbf{a}\sigma_{n-1}$ *if* $d - 1 \leq m < n$,
- $\mathbf{a}\sigma_n$ *otherwise.*

Rule for creating the 7-order shorthand universal cycle Gray code

Theorem 8 *Suppose* $\mathbf{a} \in \Pi(n)$ *where* $m = \max(a_1, a_n)$ *and* d *is the last index in its decreasing substring. The permutation of* $\mathsf{C}(n)$ *that follows* $\mathbf{a}$ *is*

- $\mathbf{a}\sigma_{n-1}$ *if* $m < n$ *and either (i)* $d = n$ *or (ii)* $d = n-1$ *and* $a_1 < a_{n-1}$,
- $\mathbf{a}\sigma_n$ *otherwise.*

Rule for creating the cool-lex shorthand universal cycle Gray code

CAT Generation of Sub-Permutations

```
visit()
for j ← 1 to m − 2
    shift(j, m − 1)
    for i ← m − 2 down to j
        Cool(i)
        a_i ↔ a_{i+1}
    end
end
```

cool-lex order

```
if m = n then
    visit()
    return
end
Bell7(m + 1)
shift(n − m, n − 1)
for i ← n − 2 down to n
    Bell7(m + 1)
    a_i ↔ a_{i+1}
end
```

7-order

```
Procedure HC(x, y)
if x = n then
    output(y)
else
    for i ← 1, 2, . . . , x do
        HC(x + 1, x + 1)
    end for
    HC(x + 1, x + 2 − y)
end if
```

Corbett
(Hamilton cycle)

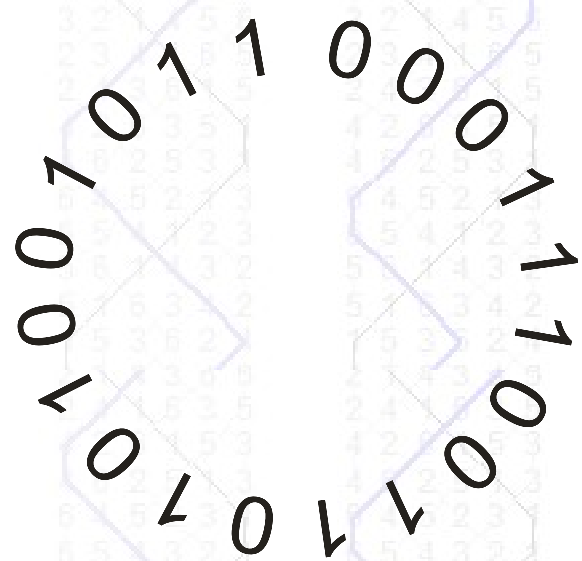
Loopless Generation of Binary Strings

$a_1 \cdots a_{n-1} \leftarrow 0 \cdots 0$
$d_1 \cdots d_{n-1} \leftarrow 1 \cdots 1$
$f_1 \cdots f_{n-1} \leftarrow 1 \cdots n-1$
loop
    $j \leftarrow f_1$
    if $a_j = 0$ **or** $a_j = n-j-1$ **then**
        output(001^{n-2})
    else if $d_j = 1$ **then**
        output$(001^{j-1}0^{n-a_j-j-1}10^{a_j-1})$
    else
        output$(001^{j-1}0^{a_j}10^{n-j-a_j-2})$
    end
    if $j = n-1$ **then**
        return
    end
    $f_1 \leftarrow 1$
    $a_j \leftarrow a_j + d_j$
    if $a_j = 0$ **or** $a_j = n-j-1$ **then**
        $d_j \leftarrow -d_j$
        $f_j \leftarrow f_{j+1}$
        $f_{j+1} \leftarrow j+1$
    end
end

Loopless algorithm for the binary representation of the shorthand universal cycle whose sub-permutations are 7-order

Open and Closed Problems

- An efficient construction for (periodic) max-weight shorthand universal cycles is open
- An efficient construction for shorthand universal cycles for the permutations of a *multiset* has been solved



Shorthand universal cycle for the permutations of {0,0,0,1,1,1}
ie a fixed-density de Bruijn cycle for n=6 and d=3