De Bruijn Sequences for the Binary Strings with Maximum Specified Density

Joe Sawada[1], Brett Stevens[2*], and Aaron Williams[2]

[1] jsawada@uoguelph.ca School of Computer Science, University of Guelph, CANADA
[2] brett@math.carleton.ca Department of Mathematics and Statistics, Carleton University, CANADA
[3] haron@uvic.ca Department of Mathematics and Statistics, Carleton University, CANADA

Abstract. A de Bruijn sequence is a circular binary string of length 2^n that contains each binary string of length n exactly once as a substring. A maximum-density de Bruijn sequence is a circular binary string of length $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$ that contains each binary string of length n with density (number of 1s) between 0 and m, inclusively. In this paper we efficiently generate maximum-density de Bruijn sequences for all values of n and m. When $n = 2m + 1$ our result gives a "complement-free de Bruijn sequence" which is a circular binary string of length 2^{n-1} that contains each binary string of length n or its complement exactly once as a substring.

Keywords: de Bruijn sequence, fixed-density de Bruijn sequence, Gray codes, necklaces, Lyndon words, cool-lex order

1 Introduction

Let $\mathbf{B}(n)$ be the set of binary strings of length n. The *density* of a binary string is its number of 1s. Let $\mathbf{B}_d(n)$ be the subset of $\mathbf{B}(n)$ whose strings have density d. Let $\mathbf{B}(n, m) = \mathbf{B}_0(n) \cup \mathbf{B}_1(n) \cup \cdots \cup \mathbf{B}_m(n)$ be subset of $\mathbf{B}(n)$ whose strings have density at most m. A *de Bruijn sequence* (or *de Bruijn cycle*) is a circular binary string of length 2^n that contains each string in $\mathbf{B}(n)$ exactly once as a substring [2]. De Bruijn sequences were introduced by de Bruijn [2] (and see earlier [3]) and have many generalizations, variations, and applications. For example, one can refer to the recently published proceedings of the *Generalizations on de Bruijn Sequences and Gray Codes* workshop [6].

In this paper we consider a new generalization of de Bruijn sequences that specifies the maximum density of the substrings. A *maximum-density de Bruijn sequence* is a binary string of length $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$ that contains each string in $\mathbf{B}(n, m)$ exactly once as a substring. For example,

$$0000011000101001$$

is a maximum-density de Bruijn sequence since its 16 substrings of length 5 (including those that "wrap-around") are precisely $\mathbf{B}(5, 3)$. Our main results are (1) an explicit construction of maximum-density de Bruijn sequences for all values of n and m, and (2) an efficient algorithm that generates these de Bruijn sequences.

We make four simple observations involving maximum-density de Bruijn sequences for $\mathbf{B}(n, m)$:

1. A maximum-density de Bruijn sequence is simply a de Bruijn sequence when $n = m$.
2. Complementing each bit in a maximum-density de Bruijn sequence results in a *minimum-density de Bruijn sequence* for the binary strings of length n whose density is at least $n - m$.
3. A maximum-density de Bruijn sequence is a*complement-free de Bruijn sequence* when $n = 2m + 1$. This is because each binary strings of length n either has density at most m or has density at least $n - m$.

* Research supported in part by NSERC

4. Reversing the order of the bits in a maximum-density de Bruijn sequence simply gives another maximum-density de Bruijn sequence for the same values of n and m. It is easier to describe our sequences in one order, and then generate them in the reverse order.

Section 2 provides background results. Section 3 describes our construction and proves its correctness. Section 4 provides an algorithm that generates our construction and analyzes its efficiency.

2 Background

A *necklace* is a binary string in its lexicographically smallest rotation. The necklaces over $\mathbf{B}(n)$ and $\mathbf{B}_d(n)$ are denoted $\mathbf{N}(n)$ and $\mathbf{N}_d(n)$, respectively. The *aperiodic prefix* of a string $\alpha = a_1a_2\cdots a_n$ is its shortest prefix $\rho(\alpha) = a_1a_2\cdots a_k$ such that $\rho(\alpha)^{n/k} = \alpha$. As is customary, the previous expression uses exponentiation to refer to repeated concatenation. Observe that if $|\rho(\alpha)| = k$, then α has k distinct rotations. For example, $\rho(0010100101) = 00101$ and 0010100101 is a necklace since it is lexicographically smaller than its other four distinct rotations 0101001010, 1010010100, 0100101001, and 1001010010. A necklace α is *aperiodic* if $\rho(\alpha) = \alpha$.

One of the most important results in the study of de Bruijn sequences is due to Fredricksen, Kessler and Maiorana [4, 5] (also see Knuth [7]). These authors proved that a de Bruijn sequence for $\mathbf{B}(n)$ can be constructed by concatenating the aperiodic prefixes of the strings in $\mathbf{N}(n)$ in lexicographic order. For example, the lexicographic order of $\mathbf{N}(6)$ is

$$000000,\ 000001,\ 000011,\ 000101,\ 000111,\ 001001,\ 001011,\\ 001101,\ 001111,\ 010101,\ 010111,\ 011011,\ 011111,\ 111111 \quad (1)$$

and so the following is a de Bruijn sequence for $\mathbf{B}(6)$, where $\cdot$ visually separates the aperiodic prefixes from (1) in the concatenation

$$0 \cdot 000001 \cdot 000011 \cdot 000101 \cdot 000111 \cdot 001 \cdot 001011 \cdot\\ 001101 \cdot 001111 \cdot 01 \cdot 010111 \cdot 011011 \cdot 011111 \cdot 1. \quad (2)$$

Although (2) is written linearly, we treat it as a circular string so that its substrings can "wrap-around" from the end to the beginning. Interestingly, (2) is also the lexicographically smallest de Bruijn sequence for $\mathbf{B}(6)$ (when written linearly). Subsequent analysis by Ruskey, Savage, and Wang [8] proved that these lexicographically smallest de Bruijn sequences can be generated efficiently for all values of n.

Recently, it was shown that this *necklace-prefix algorithm* can be modified to create a restricted type of de Bruijn sequence. *Reverse cool-lex order* is a variation of co-lexicographic order that was first defined for $\mathbf{B}_d(n)$ by Ruskey and Williams [11], and has since been generalized to subsets of $\mathbf{B}_d(n)$ including $\mathbf{N}_d(n)$ by Ruskey, Sawada, Williams [10]. In this paper we let $\mathsf{cool}_d(n)$ denote the reverse cool-lex order of $\mathbf{N}_d(n)$. The order for $n = 8$ and $d = 4$ appears below

$$\mathsf{cool}_4(8) = 00001111,\ 00011101,\ 00110101,\ 01010101,\ 00101101,\\ 00011011,\ 00110011,\ 00101011,\ 00010111,\ 00100111. \quad (3)$$

Let $\mathsf{dB}_d(n)$ denote the concatenation of the aperiodic prefixes of $\mathsf{cool}_d(n+1)$. For example, the concatenation of the aperiodic prefixes of (3) gives the following

$$\mathsf{dB}_4(7) = 00001111 \cdot 00011101 \cdot 00110101 \cdot 01 \cdot 00101101 \cdot\\ 00011011 \cdot 0011 \cdot 00101011 \cdot 00010111 \cdot 00100111. \quad (4)$$

Observe that the circular string in (4) contains each string in $\mathbf{B}_3(7) \cup \mathbf{B}_4(7)$ exactly once as a substring, and has no other substrings of length 7. For this reason we describe it as a *dual-density de Bruijn sequence* for $\mathbf{B}_3(7) \cup \mathbf{B}_4(7)$ in this paper[4]. More generally, Ruskey, Sawada, and Williams [9] proved the following result.

Theorem 1. *[9] The circular string $\mathsf{dB}_d(n)$ is a dual-density de Bruijn sequence for $\mathbf{B}_{d-1}(n) \cup \mathbf{B}_d(n)$ when $1 < d < n$.*

In this paper we do not need to completely understand the proof of Theorem 1, but we do need a simple property for its dual-density de Bruijn sequences. In other words, we need to treat each $\mathsf{dB}_d(n)$ as a "gray box". The specific property we need is stated in the following simple lemma involving reverse cool-lex order. This lemma follows immediately from equation (5.1) in [9].

Lemma 1. *[9] If $\mathbf{N}_d(n)$ contains at least three necklaces, then*

- *the first necklace in $\mathsf{cool}_d(n+1)$ is $0^{n-d+1}1^d$, and*
- *the second necklace in $\mathsf{cool}_d(n+1)$ is $0^{n-d}1^{d-1}01$, and*
- *the last necklace in $\mathsf{cool}_d(n+1)$ is $0^x10^y1^{d-1}$*

where $x = \lceil (n+1-d)/2 \rceil$ and $y = \lfloor (n+1-d)/2 \rfloor$. Moreover, each of the necklaces given above are distinct from one another and are aperiodic.

In Section 3 we will be taking apart the dual-density de Bruijn sequence $\mathsf{dB}_d(n)$ around the location of the necklace $0^{n-d+1}1^d$ from $\mathsf{cool}_d(n+1)$. For this reason we make two auxiliary definitions. Let $\mathsf{cool'}_d(n+1)$ equal $\mathsf{cool}_d(n+1)$ except that the first necklace $0^{n-d+1}1^d$ omitted. Similarly, let $\mathsf{dB'}_d(n)$ be the concatenation of the aperiodic prefixes of $\mathsf{cool'}_d(n+1)$. For example, we will be splitting $\mathsf{dB}_4(7)$ in (4) into 00001111 and

$$\mathsf{dB'}_4(7) = 00011101 \cdot 00110101 \cdot 01 \cdot 00101101 \cdot 00011011 \cdot 0011 \cdot 00101011 \cdot 00010111 \cdot 00100111$$

3 Construction

In this section we define a circular string $\mathsf{dB}(n,m)$ of length $1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$. Then we prove that $\mathsf{dB}(n,m)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,m)$ in Theorem 2.

$$\mathsf{dB}(n,m) = \begin{cases} 0\ 0^{n-1}1^2\ 0^{n-3}1^4\ \cdots\ 0^{n-m+1}1^m\ \mathsf{dB'}_m(n)\ \cdots\ \mathsf{dB'}_4(n)\ \mathsf{dB'}_2(n) & \text{if } m \text{ is even} \quad (5a) \\ 0^n1\ 0^{n-2}1^3\ 0^{n-4}1^5\ \cdots\ 0^{n-m+1}1^m\ \mathsf{dB'}_m(n)\ \cdots\ \mathsf{dB'}_5(n)\ \mathsf{dB'}_3(n) & \text{if } m \text{ is odd} \quad (5b) \end{cases}$$

Table 1 provides examples of $\mathsf{dB}(n,m)$ when $n = 7$. To understand (5), observe that $\mathsf{dB}(n,m)$ is obtained by "splicing" together the dual-density de Bruijn sequences $\mathsf{dB}_d(n+1) = 0^{n-d+1}1^d\ \mathsf{dB'}_d(n+1)$ for $d = 0, 2, 4, \ldots, m$ in (5a) or $d = 1, 3, 5, \ldots, m$ in (5b). In particular, $\mathsf{dB}_0(n+1) = 0$ is the aperiodic prefix of 0^{n+1} on the left side of (5a), and the empty $\mathsf{dB'}_0(n)$ and $\mathsf{dB'}_1(n)$ are omitted from the right sides of (5a) and (5b), respectively. For the order of the splicing, observe that $\mathsf{dB}_m(n) = 0^{n-m+1}1^m\ \mathsf{dB'}_m(n)$ appears consecutively in $\mathsf{dB}(n,m)$. That is,

$$\mathsf{dB}(n,m) = \cdots\ \mathsf{dB}_m(n)\ \cdots.$$

More specifically, $\mathsf{dB}(n,m)$ is obtained by inserting $\mathsf{dB}_m(n)$ into the portion of $\mathsf{dB}(n,m-2)$ that contains $\mathsf{dB}_{m-2}(n)$. To be precise, $\mathsf{dB}_m(n)$ is inserted between the first and second necklaces of $\mathsf{cool}_{m-2}(n+1)$. That is,

$$\mathsf{dB}(n,m) = \cdots \underbrace{0^{n-m+3}1^{m-2}}_{\text{first in } \mathsf{cool}_{m-2}(n+1)} \quad \mathsf{dB}_m(n) \quad \underbrace{0^{n-m+2}1^{m-3}01}_{\text{second in } \mathsf{cool}_{m-2}(n+1)} \quad \cdots.$$

Cool-lex orders (even densities)		Maximum-density sequences $\mathsf{dB}(7,2)$	$\mathsf{dB}(7,4)$	$\mathsf{dB}(7,6)$
0^*1^*	00000000	0	0	0
	00000011	00000011	00000011	00000011
	00001111		00001111	00001111
	00111111			00111111
$\mathsf{cool}_2(8)$	$\mathsf{cool'}_2(8)$			
00000011				
00000101	00000101	00000101	00000101	00000101
00001001	00001001	00001001	00001001	00001001
00010001	00010001	0001	0001	0001
$\mathsf{cool}_4(8)$	$\mathsf{cool'}_4(8)$			
00001111				
00011101	00011101		00011101	00011101
00110101	00110101		00110101	00110101
01010101	01010101		01	01
00101101	00101101		00101101	00101101
00011011	00011011		00011011	00011011
00110011	00110011		0011	0011
00101011	00101011		00101011	00101011
00010111	00010111		00010111	00010111
00100111	00100111		00100111	00100111
$\mathsf{cool}_6(8)$	$\mathsf{cool'}_6(8)$			
00111111				
01110111	01110111			0111
01101111	01101111			01101111
01011111	01011111			01011111

Cool-lex orders (odd densities)		Maximum-density sequences $\mathsf{dB}(7,1)$	$\mathsf{dB}(7,3)$	$\mathsf{dB}(7,5)$	$\mathsf{dB}(7,7)$
0^*1^*	00000001	00000001	00000001	00000001	00000001
	00000111		00000111	00000111	00000111
	00011111			00011111	00011111
	01111111				01111111
$\mathsf{cool}_3(8)$	$\mathsf{cool'}_2(8)$				
00000111					
00001101	00001101		00001101	00001101	00001101
00011001	00011001		00011001	00011001	00011001
00010101	00010101		00010101	00010101	00010101
00100101	00100101		00100101	00100101	00100101
00001011	00001011		00001011	00001011	00001011
00010011	00010011		00010011	00010011	00010011
$\mathsf{cool}_5(8)$	$\mathsf{cool'}_4(8)$				
00011111					
00111101	00111101			00111101	00111101
00111011	00111011			00111011	00111011
01011011	01011011			01011011	01011011
00110111	00110111			00110111	00110111
01010111	01010111			01010111	01010111
00101111	00101111			00101111	00101111
$\mathsf{cool}_7(8)$	$\mathsf{cool'}_6(8)$				
01111111					

Table 1. Maximum-density de Bruijn sequences constructed from cool-lex order of necklaces when $n = 7$ and $m > 0$. For example, $\mathsf{dB}(7,2) = 0$ 00000011 00000101 00001001 0001.

Theorem 2. *The circular string* $\mathsf{dB}(n,m)$ *is a maximum-density de Bruijn sequence for* $\mathbf{B}(n,m)$.

Proof. The claim can be verified when $n \leq 4$. The proof for $n \geq 5$ is by induction on m. The result is true when $m \in \{0,1\}$ since $\mathsf{dB}(n,0) = 0$ and $\mathsf{dB}(n,1) = 0^n1$ are maximum-density de Bruijn sequences for $\mathbf{B}(n,0)$ and $\mathbf{B}(n,1)$, respectively. The remaining base case of $m = 2$ gives $\mathsf{dB}(n,2) = 0\ 0^{n-1}11\ \mathsf{dB'}_2(n) = 0\ \mathsf{dB}_2(n)$, which is a maximum-density de Bruijn sequence for $\mathbf{B}(n,2)$ since $\mathsf{dB}_2(n)$ is a dual-density de Bruijn sequence for $\mathbf{B}_1(n) \cup \mathbf{B}_2(n)$ by Theorem 1.

First we consider the special case where $m = n$. In this case, $\mathbf{N}_{n-1}(n+1)$ contains at least three necklaces. Therefore, Lemma 1 implies that $\mathsf{dB}(n,n-2)$ and $\mathsf{dB}(n,n)$ can be expressed as follows

$$\begin{aligned} \mathsf{dB}(n,n-2) &= \cdots 0001^{n-2} \qquad\qquad 001^{n-3}01\cdots \\ \mathsf{dB}(n,n) &= \cdots 0001^{n-2} \underbrace{01^n}_{\mathsf{dB}_n(n+1)} 001^{n-3}01\cdots \end{aligned}$$

Observe that every substring of length n that appears in $\mathsf{dB}(n,n-2)$ also appears in $\mathsf{dB}(n,n)$. Furthermore, the substrings of length n that appear in $\mathsf{dB}(n,n)$ and not $\mathsf{dB}(n,n-2)$ are precisely $1^{n-2}01, 1^{n-3}011, \ldots, 01^n, 1^n, 1^{n-1}0$, which are an ordering of $\mathbf{B}_{n-1}(n) \cup \mathbf{B}_n(n)$. Since $\mathsf{dB}(n,n-2)$ is

[4] The string in (4) is described as a *fixed-density de Bruijn sequence* in [9] since each substring in $\mathbf{B}_3(7) \cup \mathbf{B}_4(7)$ can be uniquely extended to a string in $\mathbf{B}_4(8)$ by appending its 'missing' bit.

a maximum-density de Bruijn sequence for $\mathbf{B}(n, n-2)$ by induction, we have proven that $\mathsf{dB}(n,n)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,n)$.

Otherwise $m < n$. In this case, $\mathbf{N}_{m-2}(n+1)$ and $\mathbf{N}_m(n+1)$ both contain at least three necklaces. Therefore, Lemma 1 implies that $\mathsf{dB}(n, m-2)$ and $\mathsf{dB}(n,m)$ can be expressed as follows

$$\mathsf{dB}(n, m-2) = \cdots 0^{n-m+3}1^{m-2} \qquad\qquad 0^{n-m+2}1^{m-3}01 \cdots \quad (6)$$

$$\mathsf{dB}(n, m) = \cdots 0^{n-m+3}1^{m-2} \underbrace{0^{n-m+1}1^m \; 0^{n-m}1^{m-1}01 \;\cdots\; 0^x10^y1^{m-1}}_{\mathsf{dB}_m(n)} 0^{n-m+2}1^{m-3}01 \cdots \quad (7)$$

where $x = \lceil (n+1-m)/2 \rceil$, $y = \lfloor (n+1-m)/2 \rfloor$, and the bounds m and n imply that $0^x10^y1^{m-1}$ and $0^{n-m+2}1^{m-3}01$ are aperiodic. The substrings of length n in $\mathsf{dB}(n, m-2)$ are $\mathbf{B}(n, m-2)$ by induction, and the substrings of length n in $\mathsf{dB}_m(n)$ are $\mathbf{B}_{m-1}(n) \cup \mathbf{B}_m(n)$ by Theorem 1. Therefore, we can complete the induction by proving that the substrings of length n in $\mathsf{dB}(n,m)$ include those in (a) $\mathsf{dB}(n, m-2)$, and (b) $\mathsf{dB}_m(n)$. To prove (a), observe that the substrings of length n in $0^{n-m+3}1^{m-2}0^{n-m+2}$ from (6) are in $0^{n-m+3}1^{m-2}0^{n-m+1}$ from (7), except for $1^{m-2}0^{n-m+2}$. Similarly, the substrings of length n in $1^{m-2}0^{n-m+2}1^{m-3}$ from (6) are in $1^{m-1}0^{n-m+2}1^{m-3}$ from (7), and the latter also includes $1^{m-2}0^{n-m+2}$. To prove (b), consider how the insertion of $\mathsf{dB}_m(n)$ into $\mathsf{dB}(n,m)$ in (7) affects its substrings that can no longer "wrap-around" in $\mathsf{dB}_m(n)$. The substrings of length n in the wrap-around $1^{m-1}0^{n-m+1}$ in $\mathsf{dB}_m(n)$ are all in $1^{m-1}0^{n-m+2}$ from (7). Therefore, $\mathsf{dB}(n,m)$ is a maximum-density de Bruijn sequence for $\mathbf{B}(n,m)$. □

Corollary 1 follows from the first three simple observations made in Section 1.

Corollary 1. *The construction of the maximum-weight de Bruijn sequences* $\mathsf{dB}(n,m)$ *includes*

1. $\mathsf{dB}(n,n)$ *is a de Bruijn sequence for* $\mathbf{B}(n)$,
2. $\mathsf{dB}(n,m)$ *is a minimum-weight de Bruijn sequence for* $\mathbf{B}_{n-m}(n) \cup \mathbf{B}_{n-m+1}(n) \cup \cdots \cup \mathbf{B}_n(n)$,
3. $\mathsf{dB}(2m+1, m)$ *is a complement-free de Bruijn sequences for* $\mathbf{B}(2m+1)$.

4 Algorithm

As mentioned in Section 1, the *reversal* of $\mathsf{dB}(n,m)$, denoted $\mathsf{dB}(n,m)^R$ also yields a maximum-density de Bruijn sequence. To efficiently produce $\mathsf{dB}(n,m)^R$ we can use the recursive cool-lex algorithm described in [12] to produce the reversal of $\mathsf{dB}_d(n)$. In that paper, details are provided to trim each necklace to its longest Lyndon prefix and to output the strings in reverse order. An analysis shows that on average, each n bits can be visited in constant time. There are two data structures used to maintain the current necklace: a string representation $a_1a_2\cdots a_n$, and a block representation $B_cB_{c-1}\cdots B_1$ where a *block* is defined to be a maximal substring of the form 0^*1^*. A block of the form 0^s1^t is represented by (s,t). Since the number of blocks c is maintained as a global parameter, it is easy to test if the current necklace is of the form 0^*1^*: simply test if $c = 1$. By adding this test, it is a straightforward matter to produce the reversal of $\mathsf{dB}'_d(n)$. To be consistent with the description in [12], the function $\mathsf{Gen}(n-d, d)$ can be used to produce $\mathsf{dB}'_d(n)$. Using this function, the following pseudocode can be used to produce $\mathsf{dB}(n,m)^R$:

```
if m is even then  start := 2
else start := 3
for i from start by 2 to m do
      Initialize(n+1, i)
      Gen(n+1-i, i)
for i from m by 2 downto 0 do Print( 1^i 0^(n+i-1) )
if m is even then  Print( 0 )
```

The $\mathsf{Initialize}(n+1, i)$ function sets $a_1 a_2 \cdots a_{n+1}$ to $0^{n+1-i}1^i$, sets $c = 1$ and sets $B_1 = (n+1-i, i)$. The first time it is called it requires $O(n)$ time, and for each subsequent call, the updates can be performed in $O(1)$ time. Also note that the string visited by the Print() function can also be updated in constant time after the first string is visited. Since the extra work outside the calls to Gen requires $O(n)$ time, and because the number of bits in $\mathsf{dB}(n, m)$ is $\Omega(n^2)$ where $1 < m < n$, we obtain the following theorem.

Theorem 3. *The maximum-density de Bruijn sequence $\mathsf{dB}(n, m)^R$ can be generated in constant amortized time for each n bits visited, where $1 < m < n$.*

5 Open Problems

A natural open problem is to efficiently construct *density-range de Bruijn sequences* for the binary strings of length n whose density is between i and j (inclusively) for any $0 \leq i < j \leq n$.

References

1. F. Chung, P. Diaconis, and R. Graham, *Universal cycles for combinatorial structures*, Discrete Mathematics, 110 (1992) 43–59.
2. N.G. de Bruijn, *A combinatorial problem*, Koninkl. Nederl. Acad. Wetensch. Proc. Ser A, 49 (1946) 758–764.
3. N.G. de Bruijn, *Acknowledgement of priority to C. Flye Sainte-Marie on the counting of circular arrangements of 2n zeros and ones that show each n-letter word exactly once*, T.H. Report 75-WSK-06, Technological University Eindhoven (1975) 13 pages.
4. H. Fredericksen and J. Maiorana, *Necklaces of beads in k colors and kary de Bruijn sequences*, Discrete Mathematics, 23 3 (1978) 207–210.
5. H. Fredericksen and I. J. Kessler, *An algorithm for generating necklaces of beads in two colors*, Discrete Mathematics, 61 (1986) 181–188.
6. G. Hurlbert, B. Jackson, B. Stevens (Editors), *Generalisations of de Bruijn sequences and Gray codes*, Discrete Mathematics, 309 (2009) 5255–5348.
7. D.E. Knuth, *The Art of Computer Programming, Volume 4, Generating all tuples and permutations*, Fascicle 2, Addison-Wesley, 2005.
8. F. Ruskey, C. Savage, and T.M.Y. Wang, *Generating necklaces*, J. Algorithms, 13 (1992) 414–430.
9. F. Ruskey, J. Sawada, and A. Williams *Fixed-density de Bruijn sequences*, (submitted) 2010.
10. F. Ruskey, J. Sawada, and A. Williams *Binary bubble languages and cool-lex order*, (submitted) 2010.
11. F. Ruskey and A. Williams, *The coolest way to generate combinations*, Discrete Mathematics, 17 309 (2009) 5305–5320.
12. J. Sawada and A. Williams *A Gray Code for fixed-density necklace and Lyndon words in constant amortized time*, (submitted) 2010.